MOVING LOADS ON RAILWAY UNDERBRIDGES

H. BAMFORD

MOVING LOADS ON RAILWAY UNDERBRIDGES.

MOVING LOADS ON RAILWAY UNDERBRIDGES

INCLUDING:

DIAGRAMS OF BENDING MOMENTS AND SHEARING FORCES

AND

TABLES OF EQUIVALENT UNIFORM LIVE LOADS.

BY

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WHITTAKER & CO.,

2, White Hart Street, Paternoster Square, London, E.C.;

and 64–66, Fifth Avenue, New York.

1907.

LONDON:
PRINTED BY WILLIAM CLOWES AND SONS, LIMITED,
DUKE STREET, STAMFORD STREET, S.E., AND GREAT WINDMILL STREET, W.

PREFACE.

THE attention of the author having been directed, some five years ago, to the tedious and unscientific methods commonly used in practice in the preparation of tables of "equivalent uniform live loads" for railway under-line bridges, it occurred to him that much of the labour which the application of these methods involved might be saved, and the work simplified, by a direct application of the funicular polygon. Working on these lines he has succeeded in devising a graphical method, whereby, on a single diagram, the maximum shears and the maximum bending moments and the points along the spans at which they occur can be determined with facility for a wide range of spans and for any given type-train. A full description of this method originally appeared in *Engineering* on September 7th, 1906, and was followed in quick succession by three other articles dealing more or less closely with the same subject. These four articles are now brought together in Chapters IV–VII of the following work, and three other chapters on Diagrams of Bending Moments and Shearing Forces in Beams added, in the hope that it may prove useful to engineering students in general and to designers of railway underbridges in particular.

The author's thanks are due to *Engineering* for permission to reproduce the articles above referred to and for kindly supplying electros of the diagrams.

H. BAMFORD.

Glasgow, November 1907.

CONTENTS.

MOVING LOADS ON RAILWAY UNDERBRIDGES.

CHAPTER I.

Bending Moments and Shearing Forces in Beams.

The bending moment and shearing force at any cross section, or point, as it is more conveniently termed, of a beam, are generally determined on the assumption that the external forces acting upon the beam are all in one plane—the plane of bending—and at right angles to the direction of the beam's length. Now, for any such system of co-planer forces, the conditions of equilibrium are:—

(1) The algebraical sum of the components of the forces taken in any direction is nil; and

(2) The algebraical sum of the moments of the forces about any point in their plane is nil. If the forces under consideration, however, be not in equilibrium, then:

(3) The algebraical sum of the components of the forces taken in any direction is equal to the component of their resultant taken in the same direction; and

(4) The algebraical sum of the moments of the forces about any point in their plane is equal to the moment of their resultant about the same point.

Let A B, therefore, Fig. 1, represent a beam of span L supported at both ends and loaded with a single load W concentrated at a point E, distant d from

B, and let P_A and P_B be the supporting forces at A and B respectively. Then, from the above conditions of equilibrium, we have:

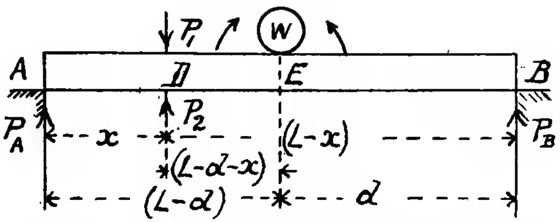
$$P_A + P_B - W = 0; \text{ or } P_A + P_B = W;$$

and, taking moments about B and A respectively,

$$P_A . L - W . d = 0; \text{ or } P_A = \frac{W . d}{L},$$

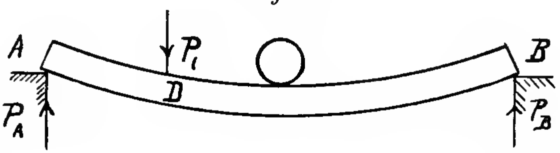
and $$W (L - d) - P_B . L = 0; \text{ or } P_B = W - \frac{W . d}{L} = W - P_A.$$

Fig. 1.



Now take any point D between A and E, and consider the equilibrium of the portion A D of the beam. If, at D, we imagine two opposite forces, P_1 and P_2, each equal and parallel to P_A, to be applied, these forces, balancing one another, will not disturb the equilibrium in any way; P_A and P_1, however, will constitute a couple of moment $P_A . AD = P_A . x$, tending to rotate A D in a

Fig. 2.



clockwise direction, while the unbalanced force P_2 will tend to translate A D upwards. Considering, in the same way, the equilibrium of the portion B D, we see that P_B at B, and W at E, are equivalent to a force $= W - P_B = P_A$ acting downwards at D, together with a couple of moment $= P_B . BD - W . ED$

$$= P_B (L - x) - W (L - x - d) = -(W - P_B)(L - x) + P_A . L = P_A . x,$$

tending to rotate B D in a counter-clockwise direction.

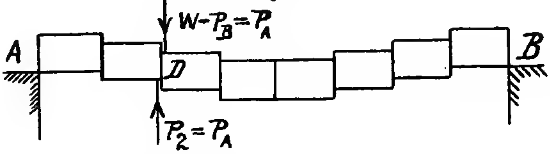
The two equal couples acting in opposite directions upon A D and B D respectively tend to bend the beam at D, and the two equal and opposite forces tend to shear the beam at the point. The moment of either couple is called the bending moment at D, and either force is called the shearing force at the point.

Again, let the beam be loaded in any manner, and assume R_1 and R_2 to be the total loads upon the portions A D and B D respectively, and a and b the distances of their centres of gravity from D (Fig. 4). Then, from the above condition of equilibrium, it follows that:

Bending moment at $D = P_A . x - R_1 . a = P_B (L - x) - R_2 . b$,

and shearing force at $D = P_A - R_1 = R_2 - P_B$.

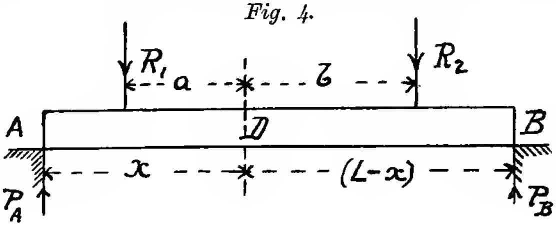
Fig. 3.



Or the bending moment at any point of a beam is the algebraical sum of the moments about the point of all the external forces acting upon the portion on either side of it, and the shearing force at the point is the algebraical sum of the external forces acting either upon the portion to the left, or upon the portion to the right of the point.

For convenience we shall regard the bending moment at any point of the

Fig. 4.



beam as positive or negative according as the beam, under the action of the external forces, tends to bend concavely upwards (as in Fig. 2), or concavely downwards at the point, and the shearing force at any point as positive or negative, according as the portion on the left of the point tends to slide upwards or downwards relatively to the portion on the right (Fig. 3).

CHAPTER II.

DIAGRAMS OF BENDING MOMENTS AND SHEARING FORCES DUE TO FIXED LOADS.

Case I.—Cantilever fixed at one end and loaded at the other.

Let A B, Fig. 5, represent a cantilever fixed at B and loaded with a single concentrated load W at A. The bending moment at any point D, distant x from A, will be $M_x = -W.x$, and at B it will be $M_L = -W.L$, and the shearing force will be constant throughout and $= -W$. The diagram of

Fig. 5.

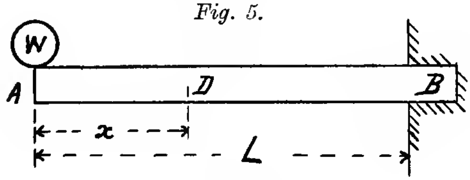


Fig. 6.

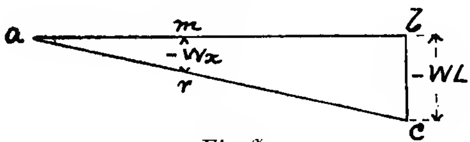
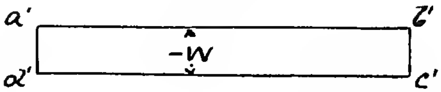


Fig. 7.



bending moments, therefore, will be a triangle, *a b c*, Fig. 6, and the diagram of shearing forces a rectangle, as shown in Fig. 7.

Case II.—Cantilever loaded with several concentrated loads.

Let the cantilever represented in Fig. 8 be loaded with the concentrated loads W_1, W_2 and W_3.

Proceeding as in Case I, the diagram of bending moments due to W_1 acting alone will be the triangle $a\,b\,b_1$, Fig. 9, and similarly the diagrams of bending moments due to W_2 and W_3 respectively will be the triangles $c\,b_1 b_2$ and $d\,b_2 b_3$, where $b_1 b_2$ and $b_2 b_3$ represent $-W_2\,(a_2 + a_3)$ and $-W_3 . a_3$ respectively.

The diagrams of shears due to W_1, W_2 and W_3 respectively will be rectangles, as shown by $a'b'c'd'$, $e'c'f'g'$ and $h'f'i'k'$, Fig. 10.

Fig. 8.

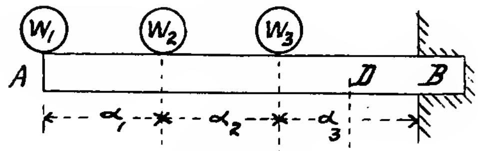


Fig. 9.

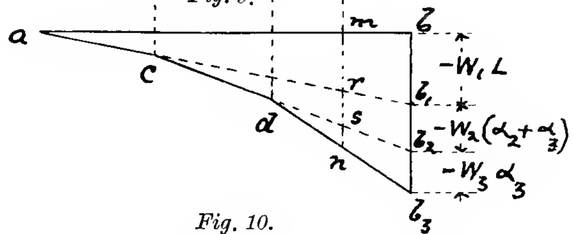
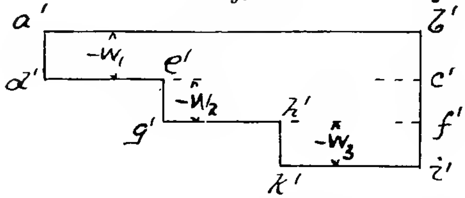


Fig. 10.



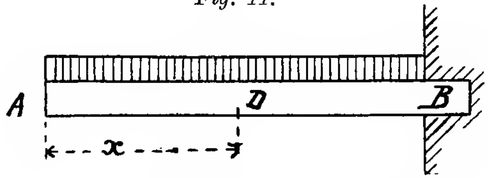
Hence, due to W_1, W_2 and W_3 combined, the diagram of bending moments will be the polygon $a\,b\,b_3 d\,c\,a$, Fig. 9, and the diagram of shears the stepped figure shown in Fig. 10.

Case III.—Cantilever loaded throughout its length with a uniformly-distributed load.

Let w be the load per unit of length, and D a point distant x from A, Fig. 11; then, since the algebraical sum of the forces acting on the portion A D

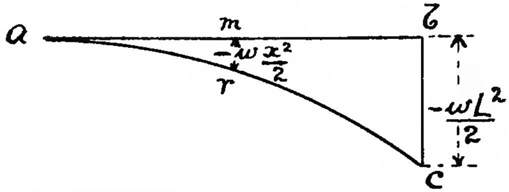
is $-wx$, and the distance from D of their centre of gravity $\frac{x}{2}$, the shear at D is $S_x = -w.x$ and the bending moment, $M_x = -w.\frac{x^2}{2}$.

Fig. 11.



The diagram of shears, therefore, is a triangle, $a'b'c'$, Fig. 13, and the diagram of bending moments the figure $a\,b\,c$, Fig. 12, where $a\,r\,c$ is a parabola having its vertex at a.

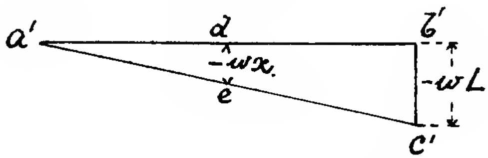
Fig. 12.



Case IV.—Cantilever loaded with a uniformly-distributed load over a portion of its length only.

Let the loaded portion A E be of length l, and w the load per unit of length;

Fig. 13



then, for this portion, the curves of bending moment and shear will be the same as in Case III, viz., a parabola ($a\,f$, Fig. 15) and straight line ($a'f'$, Fig. 16) respectively.

Between E and B the shear will be constant and $= -w.l$, and the bending

moment will increase (numerically) at a uniform rate, as shown by the straight line fc, from $-w\frac{l^2}{2}$ at E to $-w\,l\left(L-\frac{l}{2}\right)$ at B; and the bending moment at any point D, distant x from A, will be $M_x = -w.l\left(x-\frac{l}{2}\right)$.

Case V.—Beam supported at both ends and loaded with a single concentrated load.

Fig. 14.

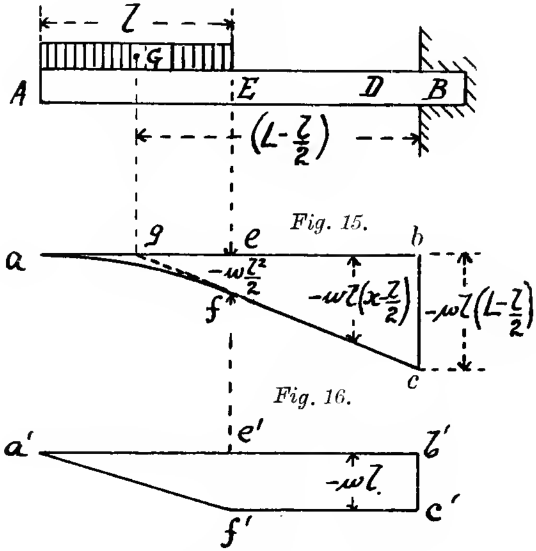


Fig. 15.

Fig. 16.

Let A B, Fig. 17, be the beam and W the load acting at a point E, distant d from B; then:

$$P_A = \frac{W.d}{L} \text{ and } P_B = \frac{W\,(L-d)}{L}.$$

At any point D, distant x from A, the bending moment will be $M_x = P_A.x$ $= \frac{W.d}{L}.x$, if D lies between A and E, or $M_x = P_A\,x - W\,(x-(L-d))$

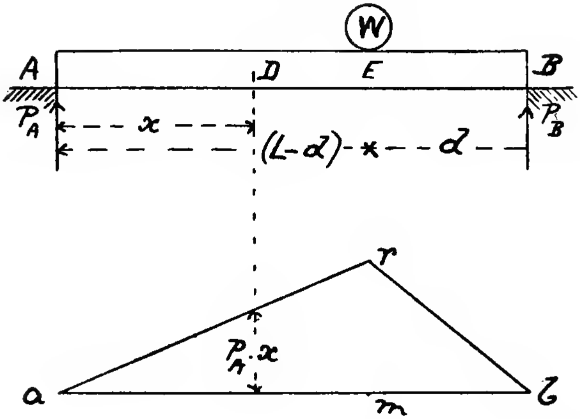
$$= \frac{W.(L-d)}{L}.(L-x) = P_B.(L-x)$$

if D lies between E and B. The bending moment diagram, therefore, will be a

triangle, $a\,r\,b$, Fig. 18, where $m\,r$ represents $P_A.(L - d) = P_B.d$, the bending moment at E.

Again, the shear for all points between A and E will be $= P_A$, and for all

Figs. 17 and 18.



points between E and B it will be $= P_A - W = - P_B$, and the diagram of shears will consist of two rectangles as shown in Fig. 19.

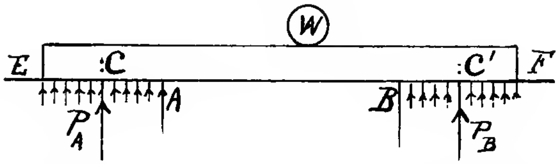
Case Va.—Let the beam in the last case extend over the supports by amounts represented by A E and B F respectively (Fig. 17a), and assume the supporting forces P_A and P_B uniformly distributed over A E and B F. The intensity (or

Fig. 19.

load per unit of length) of these loads will be $w_1 = \dfrac{P_A}{A\,E}$ and $w_2 = \dfrac{P_B}{B\,F}$, and the resultants of the loads will pass through the centres, c and c', of A E and B F. The diagram of bending moments, therefore, will be as shown in Fig. 18a, where $c\,k$ and $f\,m$ are parabolas (the same as in Case II, but inverted), and

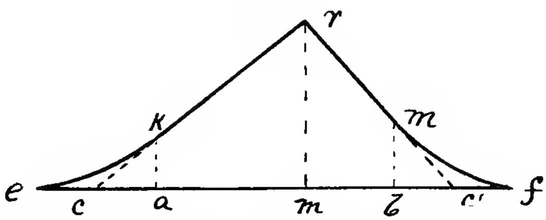
$r\,k$ and $r\,m$ straight lines passing, when produced, through c and c' respectively. The bending moment represented by $m\,r$, therefore, will be $= P_A \cdot c\,m =$

Fig. 17a.



$P_B . c'm$, and the bending moment at any point between the supports will be the same as for a beam having a span equal to $c\,c'$, with the supporting forces concentrated at the points c and c'. The distance $c\,c'$, therefore, may, so far as

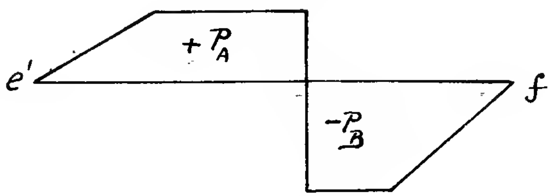
Fig. 18a.



bending moments are concerned, be called the "effective span," and A B the clear span.

Again, from the diagram of shearing forces represented in Fig. 19*a*, it is

Fig. 19a.



evident that so far as shearing forces between A and B are concerned, the supporting forces P_A and P_B may be regarded as acting at c and c' respectively or the distance $c\,c'$ will be the effective span in this case also.

The span referred to in the following examples is the effective span.

Case VI.—Beam supported at both ends, and loaded with two or more concentrated loads. Let the beam A B, Fig. 20, be acted upon by loads W_1, W_2 and W_3 at points E, F and G, distant d_1, d_2 and d_3 respectively from B. Then :—

$$P_A = \frac{W_1 . d_1 + W_2 . d_2 + W_3 . d_3}{L}, \text{ and}$$

$$P_B = \frac{W_1 (L - d_1) + W_2 (L - d_2) + W_3 (L - d_3)}{L}.$$

Now draw, under A B, the horizontal line $a\,b$, Fig. 21, to represent the span, and from b erect the perpendicular $b\,e$ to represent, on any scale, $P_A . L$, and mark the points f and g so that $e\,f$ and $f\,g$ represent, on the same scale, $W_1 . d_1$ and $W_2 . d_2$ respectively. The lines $a\,e$, $1\,f$ and $2\,g$ will then cut the verticals through E, F and G respectively in points 1, 2 and 3, such that the polygon a 1 2 3 b will be a diagram of bending moments. For if, through any point D in the beam, a vertical line be drawn cutting the lines $a\,e$, $1\,f$, $2\,g$ and $a\,b$, in r, s, n and m respectively,

$$\frac{m\,r}{am} = \frac{b\,e}{L} = P_A, \text{ or } m\,r = P_A . a\,m = P_A . A\,D;$$

and similarly, $r\,s = W_1 . E\,D$, $s\,n = W_2 . F\,D$, and $m\,n = m\,r - r\,s - s\,n$ $= P . A\,D - W_1 . E\,D - W_2 . F\,D =$ the bending moment at D. The ordinate, $m\,n$, therefore, of the polygon a 1 2 3 b, at any point m along the span, represents the bending moment at that point; or, in other words, the polygon is a diagram of bending moments.

The shearing force between A and E is constant and $= P_A$; between E and F it is $= P_A - W_1$, between F and G it is $= P_A - W_1 - W_2$, and between G and B it is $= P_A - W_1 - W_2 - W_3 = - P_B$. The diagram of shearing forces, therefore, will be as shown in Fig. 24.

Another method. A more purely graphical method than the above, and one which is of somewhat greater importance in consequence of its greater applicability and usefulness, is what may be called "the funicular polygon method," which may be described as follows.

Along a vertical line of loads a 1–3 b, Fig. 23, lay off the distances

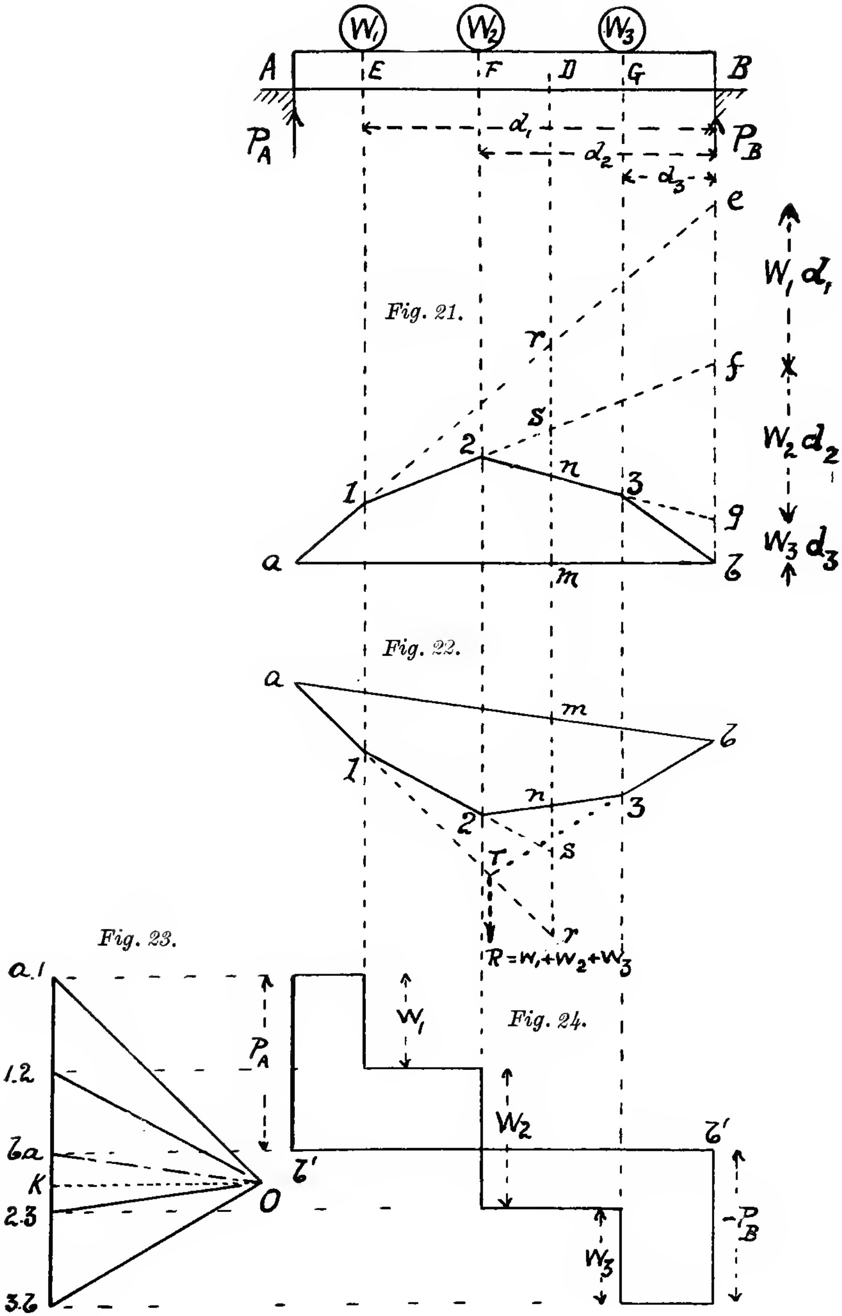
Fig. 20.

Fig. 21.

Fig. 22.

Fig. 23.

Fig. 24.



$a.1$–1.2, 1.2–2.3 and 2.3–$3\,b$, to represent, on any scale, W_1, W_2 and W_3 respectively; and taking any pole O, draw the radius vectors O–$a.1$, O–1.2, O–2.3 and O–$3.b$. Then, starting from any point a in the vertical through A, Fig. 22, draw $a\,1$, $1\,2$, $2\,3$ and $3\,b$ parallel respectively to the above radius vectors, to cut the verticals through W_1, W_2, W_3 and B (Fig. 20) in the points 1, 2, 3 and b, and then draw the radius vector O–$b\,a$ parallel to the closing line, $b\,a$, of the funicular polygon $a\,1\,2\,3\,b$. P_B and P_A will then be represented by $3.b$–$b.a$ and $b.a$–$a.1$ (Fig. 23) respectively, and the polygon $a\,1\,2\,3\,b\,a$ will be a diagram of bending moments.

For suppose $a\,1\,2\,3\,b$ to be a frame of bars supported vertically at a and b and acted upon by the loads W_1, W_2 and W_3 suspended from the joints 1, 2 and 3 respectively, the triangle of forces for joint 1 would then be $a.1$–1.2–O (Fig. 23), and that for joint a, $a.1$–O–$b.a$. Hence, if $a.1$–1.2 represent W_1, O–$a.1$ would represent the pull in $1\,a$ (Fig. 22) and $b.a$–$a.1$ the supporting force at a; and as the supporting forces are obviously the same for both the frame and beam, $b.a$–$a.1$ will represent P_A. Similarly, $3\,b$–$b\,a$ will represent P_B.

Now draw through the pole O a horizontal line to cut the line of loads in K, and through any point D in the beam a vertical line cutting $a\,1$, $1\,2$, $2\,3$ (produced if necessary) and $a\,b$ in r, s, n, and m respectively. Then, since the triangles $r\,m\,a$, $r\,s\,1$ and $s\,n\,2$, Fig. 22, are respectively similar to the triangles $b.a$–$a.1$–O, $a.1$–1.2–O and 1.2–2.3–O, Fig. 23,

$$\frac{r\,m}{b.a\text{–}a.1} = \frac{r\,m}{P_A} = \frac{r\,a}{a.1\text{–O}} = \frac{\mathrm{A\,D}}{\mathrm{O\,K}}, \text{ or } r\,m = \frac{P_A.\mathrm{A\,D}}{\mathrm{O\,K}}$$

$$\frac{r\,s}{a.1\text{–}1.2} = \frac{r\,s}{W_1} = \frac{s\,1}{1.2\text{–O}} = \frac{\mathrm{E\,D}}{\mathrm{O\,K}}, \text{ or } r\,s = \frac{W_1.\mathrm{E\,D}}{\mathrm{O\,K}}$$

and

$$\frac{s\,n}{1.2\text{–}2.3} = \frac{s\,n}{W_2} = \frac{n\,2}{2.3\text{–O}} = \frac{\mathrm{F\,D}}{\mathrm{O\,K}}, \text{ or } s\,n = \frac{W_2.\mathrm{F\,D}}{\mathrm{O\,K}}$$

Therefore $m\,n = r\,m - r\,s - s\,n = \dfrac{P_A.\mathrm{A\,D} - W_1.\mathrm{E\,D} - W_2.\mathrm{F\,D}}{\mathrm{O\,K}} = \dfrac{M x}{\mathrm{O\,K}}$,

and M_x, the bending moment at D, is $= m\,n \times \mathrm{O\,K}$, where $m\,n$ is measured on the linear scale on which A B represents the span, and O K on the scale of loads.

The diagram of shearing forces can be readily constructed by drawing the base line $b'b'$ through the point $b\,a$, Fig. 23, and projecting across from the points of division in the line of loads as shown in Fig. 24.

Note.—If the first and last sides ($a\,1$ and $b\,3$) of the funicular polygon $a\,1\,2\,3\,b$ be produced to meet in T, and $a\,\mathrm{T}\,b$ be regarded as a frame supported at a and b, it will be clear, from the triangle of forces $a.1$–$3\,b$–O, that a load $= W_1 + W_2 + W_3$ at T would give the same supporting forces at A and B as before. Hence $W_1 + W_2 + W_3$ at T is equivalent to W_1 at E $+ W_2$ at F $+$

Fig. 25.

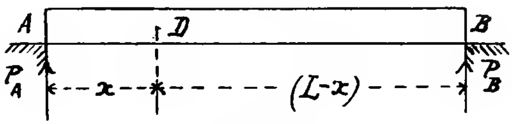


Fig. 26.

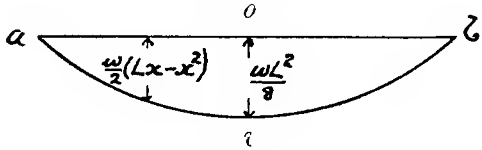
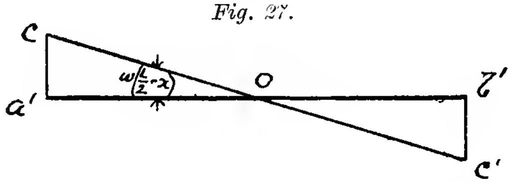


Fig. 27.



W_3 at G; or T is a point in the line of action of the resultant of W_1, W_2 and W_3.

Case VII.—Beam supported at both ends and loaded throughout its length with a uniformly-distributed load.

Let w be the load per unit of length and P_A and P_B the supporting forces. Then:—

$$P_A = P_B = \frac{w.L}{2}$$

the bending moment at any point D, distant x from A, is :—

$$M_x = P_A . x - w . \frac{x^2}{2} = \frac{w}{2}(L x - x^2) \quad . \quad . \quad (1)$$

and the shear at D is :—

$$S_x = P_A - w x = w\left(\frac{L}{2} - x\right) \quad . \quad . \quad . \quad (2)$$

The bending moment, M_x, varies with the ordinates to a parabola, $a\ v\ b$, Fig. 26, of which (1) is the equation. At the centre the bending moment is a maximum, is $= \frac{w}{2}\left(L . \frac{L}{2} - \left(\frac{L}{2}\right)^2\right) = \frac{w L^2}{8}$, and is represented by the ordinate $v\ o$ in Fig. 26. The shear, S_x, varies with the ordinates to a straight line C O C′,

Fig. 28.

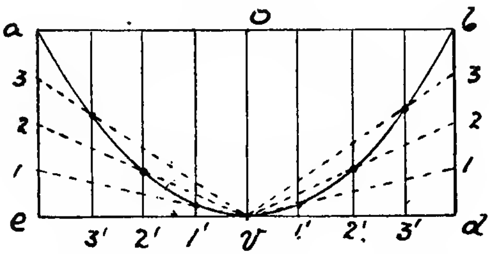


Fig. 27, from $+ \frac{w L}{2}$ at A to $- \frac{w L}{2}$ at B, and is nil at the centre.

The parabola $a\ v\ b$, Fig. 26, may be constructed as follows. Having set off $o\ v$ at right angles to $a\ b$ to represent, on any scale, $\frac{w L^2}{8}$, complete the rectangle, $a\ b\ d\ e$, Fig. 28, and divide $v\ d$, $v\ e$, $d\ b$ and $e\ a$ into the same (any) number of equal parts. Now draw through the points of division 1′, 2′, 3′ . . . in $v\ d$ and $v\ e$ lines parallel to $v\ o$ and join v to the points of division 1, 2, 3 . . . in $e\ a$ and $d\ b$. Then the intersections of $v\,1$, $v\,2$, $v\,3$ . . . with the lines drawn through 1′, 2′, 3′ . . . respectively, will give points on the parabola required.

Case VIII.—Beam supported at the ends and loaded with a uniformly-distributed load over a portion of its length only.

Let L be the length of the span; l the length of E F, the portion loaded with a uniformly-distributed load of w per unit of length, and d, the distance of F from B. Then:—

$$P_A = \frac{w\,l\left(\frac{l}{2} + d\right)}{L}$$

and

$$P_B = \frac{w\,l\left(L - d - \frac{l}{2}\right)}{L}.$$

For any point between A and E, distant x from A, $M_x = P_A\,x$ which varies with x according to the ordinates to a straight line, $a\,e$, where $e\,e'$ represents $P_A . A\,E$.

For any point, D, between E and F,

$$M_x = P_A . A\,D - W . D\,E . \frac{D\,E}{2} \text{ and at F, } M_x = P_A . A\,F - \frac{W . (E\,F)^2}{2}.$$

For points between F and B,

$$M_x = P_B\,(L - x).$$

To construct the bending moment diagram, therefore, lay off $b\,b'$, Fig. 30, to represent $P_A . L$, and let $a\,b'$ cut the vertical drawn through E in e; $e\,e'$ will then represent $P_A . A\,E$, the bending moment at E. Now let the vertical drawn from any point D, in E F, cut $a\,b$ in m and $a\,b'$ in q; then $q\,m$ will represent $P_A . A\,D$, and if we lay off along $q\,m$, a length $q\,p$ to represent $\frac{W . (D\,E)^2}{2}$, $m\,p$ will represent $P_A . A\,D - W . \frac{(D\,E)^2}{2}$, the bending moment at D. Other points may be obtained on the parabolic curve, $e\,p\,f$, in the same way, and the diagram completed by drawing the straight line $f\,b$.

Another method. Consider E F as if it were a uniformly-loaded beam supported at E and F and draw the parabolic curve $e' s f'$ for the uniform load. The bending moment at any point D of this beam will be $M_D = w\frac{E\,F}{2} . E\,D - \frac{w\,(E\,D)^2}{2}$, and will be represented by the ordinate $m\,s$. Now join e with f, Fig. 30, and let $e\,f$ cut $q\,m$ in n; then, from similar triangles

$$\frac{q\,n}{r\,f} = \frac{e\,n}{e\,f} = \frac{E\,D}{E\,F}, \text{ or } q\,n = r\,f . \frac{E\,D}{E\,F}.$$

And since rf and qp represent $\frac{w(\mathrm{EF})^2}{2}$ and $\frac{w.(\mathrm{ED})^2}{2}$ respectively, $pn\,(=qn-qp)$ represents $\frac{w.(\mathrm{EF})^2}{2}\,\frac{\mathrm{ED}}{\mathrm{EF}}-\frac{w.(\mathrm{ED})^2}{2}=\mathrm{M_D}$, or $pn=ms$.

Fig. 29.

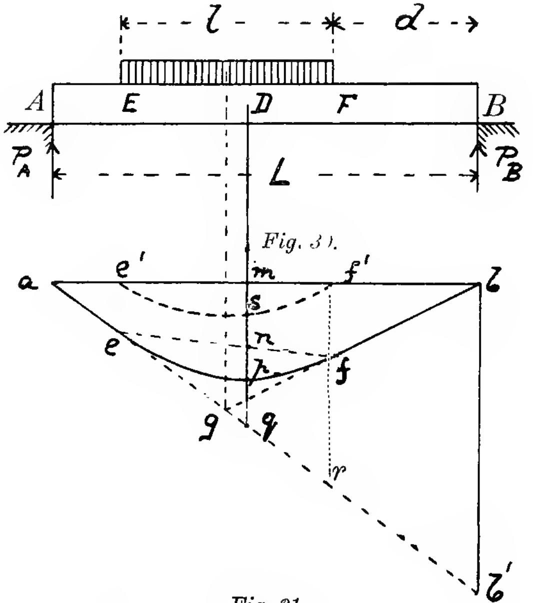
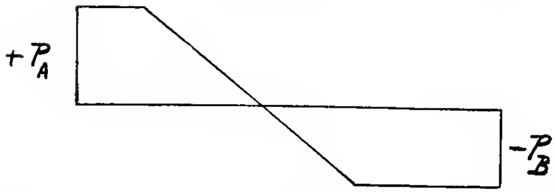


Fig. 30.

Fig. 31.



Hence, erecting the ordinates $e'e$ and $f'f$ to represent $\mathrm{P_A}.\mathrm{AE}$ and $\mathrm{P_B}.\mathrm{BF}$ (the bending moments at E and F respectively), and drawing the straight lines ae, ef and fb, and then erecting ordinates from ef equal to the corresponding ordinates of the parabola $e'sf'$, the diagram of bending moments $aepfb$ will be obtained.

The diagram of shears will be as shown in Fig. 31.

3. Relation between the bending moment and shearing force at any point of a loaded beam.

(*a*) Concentrated loads.

Let the beam be supported at the ends and loaded as represented in Fig. 20 then:

between A and E the shearing force $S_1 = P_A$;

„ E „ F „ $S_2 = P_A - W_1$;

„ F „ G „ $S_3 = P_A - W_1 - W_2$;

„ G „ B „ $S_4 = P_A - W_1 - W_2 - W_3 = - P_B$;

and at E the bending moment $M_E = P_A . AE = S_1 . AE$;

„ F „ $M_F = P_A . AF - W_1 . EF$;

$= P_A . AE + (P_A - W_1) EF = M_E + S_2 . EF$;

„ G „ $M_G = P_A . AG - W_1 EG - W_2 . FG$;

$= P_A . AF - W_1 . EF + (P_A - W_1 - W_2) FG$;

$= M_F + S_3 . FG$.

Hence $\frac{M_E}{AE} = S_1$, $\frac{M_F - M_E}{EF} = S_2$ and $\frac{M_G - M_F}{FG} = S_3$; and, in general, if S be the shearing force in the segment between any two consecutive loads, δL the length of the segment, and δM the increment of the bending moment, then: $S = \frac{\delta M}{\delta L}$, or the shear at any point of the beam is equal to the rate of increase of the bending moment at the point.

(*b*) Continuous loads.

Let the intensity of the load vary from point to point with the ordinates to a continuous curve (or straight line), as represented in Fig. 32, and consider any two points a and b at an infinitely small distance dx apart. Let M_a and S_a be the bending moment and shear respectively at a, and M_b and S_b those at b, and let w be the intensity of the load—load per unit of length—over ab, then:

$$S_b = S_a - w . dx, \text{ or } \frac{S_b - S_a}{dx} = \frac{dS}{dx} = - w \quad . \quad . \quad . \quad (1)$$

and, since the forces acting upon the portion of the beam to the left of a are equivalent to a couple M_a and a force S_a, acting upwards at a,

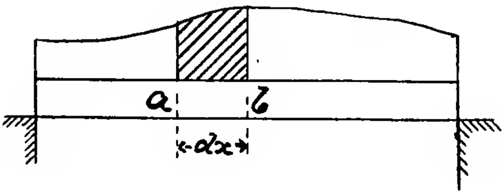
$$M_b = M_a + S_a \, d\,x - w.d\,x.\frac{d\,x}{2};$$

$$= M_a + S_a.d\,x - \text{an infinitely small quantity of the second order which may be neglected;}$$

or
$$\frac{M_b - M_a}{d\,x} = S_a = \frac{d\,M}{d\,x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, from (1), the intensity of the load at any point of the beam is equal to the rate of *decrease* of the shear at the point; and, from (2), the shear at any point is equal to the rate of *increase* of the bending moment at the point.

Fig. 32.



Again, from (1), we see that if for any portion $a\,b$ of the beam $w = 0$, then $d\,S = 0$, or the shear S is constant for that portion; and, from (2), if the shear S at any point is nil, then $\frac{d\,M}{d\,x} = 0$, or the bending moment at the point is either a maximum or a minimum.

Further, if $w = f(x)$ be the equation to the curve of loads, then $S = -\int f(x)\,d\,x$ will be the equation to the curve of shears, and $M = \int S.d\,x$ the equation to the curve of bending moments.

Example.—Let the equation to the curve of loads be $w = a\,x$, where a is a constant; then: the intensity of the load at B will be a L, the diagram of loads a triangle, A B B′, Fig. 33, the total load W upon the beam $\frac{a\,L}{2}.L$, and the distance of the centre of gravity of the load from B will be $\frac{L}{3}$.

Therefore $$P_A = \frac{\frac{a\,L}{2}.L.\frac{L}{3}}{L} = \frac{a\,L^2}{6} = \frac{W}{3}. \quad P_B = \frac{2}{3}W,$$

Fig. 33.

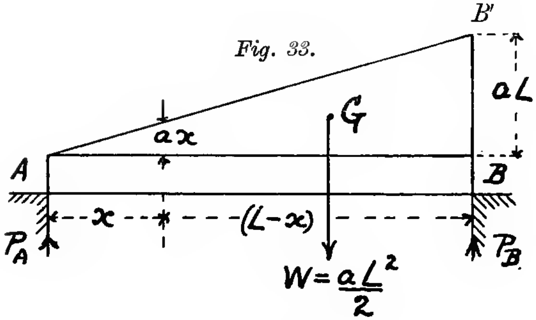


Fig. 34.

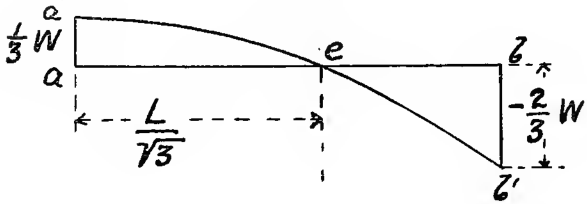
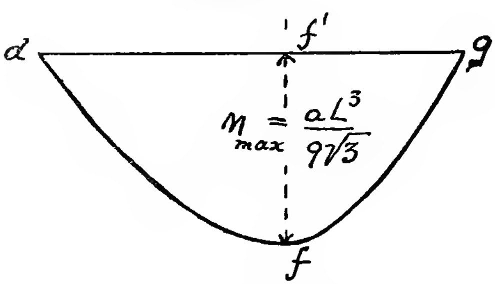


Fig. 35.



and $S = -\int f(x)\,dx = -\int a\,x.dx = -\frac{a\,x^2}{2} + c$, where c, the constant of integration, is $= P_A = \frac{a\,L^2}{6}$, since $S = P_A$ when $x = 0$. Therefore

$S = a\left(\frac{L^2}{6} - \frac{x^2}{2}\right)$, a parabola $a'eb'$, Fig. 34, of which $S = 0$ when $x = \frac{L}{\sqrt{3}} = ae$.

Again $M = \int S.dx = \int a\left(\frac{L^2}{6} - \frac{x^2}{2}\right) dx = \frac{a\,(L^2x - x^3)}{6} + 0$ (the constant of integration being nil, since $M = 0$ when $x = 0$).

Hence, $M = \frac{a}{6}(L^2x - x^3)$ is the equation to the curve dfg of bending moments (Fig. 35), and M is a maximum when $x = \frac{L}{\sqrt{3}}$, or $M_{max} = \frac{a}{6}\left(L^2.\frac{L}{\sqrt{3}} - \left(\frac{L}{\sqrt{3}}\right)^3\right) = \frac{a\,L^3}{9\sqrt{3}}$.

CHAPTER III.

Diagrams of Bending Moments and Shearing Forces due to Moving Loads.

Continuous Loads.

Case I.—Beam supported at both ends acted upon by a uniform moving load of length greater than the span.

Let w be the load per unit of length, and D a point in the beam distant x from A, Fig. 36. Assuming the load moving across the span from right to left, when the front, E, of the load is to the right of D, the bending moment at this point will be $M_x = P_A . x$, which is obviously greater when D B is fully loaded than when only partially loaded. Now suppose the load to extend up to or beyond D; then, the bending moment at D will be $M_x = P_B . (L - x) - \frac{w (L - x)^2}{2}$, and as this is a maximum for any given value of x when P_B is a maximum, it follows that the bending moment at any point D is a maximum when A B is fully loaded. The diagram of maximum bending moments, therefore, is a parabola $a\,v\,b$, Fig. 37, the equation to which is:—

$$M_x = \frac{w}{2} (Lx - x^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

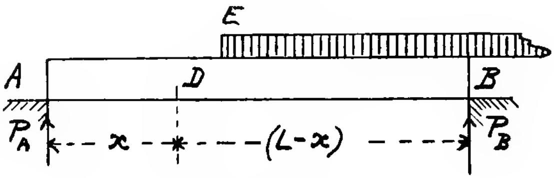
or the same as for Case VII, Chap. II.

Maximum Shear.—When the load extends over a portion, B E, only of the beam, the shearing force at any point D, between A and E, will be positive and equal to P_A; and will obviously be greater, therefore, when D B is fully loaded than when only partially loaded.

Now suppose the load to advance a little towards the left, so that the front,

E, comes to some point, E′, between D and A, and let w' be the load upon the portion D E′. As the front of the load advances from D to E′ the supporting force P_A will be increased by a fractional part only of the load w', but the shear, being equal to the new supporting force at A minus the whole of w', will be decreased. The positive shear, therefore, at any point D of the beam will be a

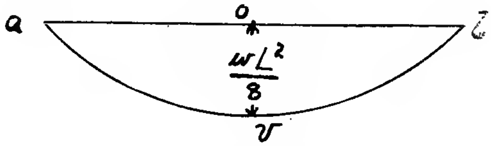
Fig. 36.



maximum when the front of the load has advanced up to the point from the right abutment B.

In the same way it is easily shown that if the load advances from A towards B the shear at any point D between B and the front of the load will be negative, and (numerically) a maximum for that point when the front of the load has advanced just up to the point.

Fig. 37.



Hence, the maximum positive shear at any point D, distant x from A, is:

$$S_x = P_A = \frac{w.(L - x)^2}{2L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the maximum negative shear is:

$$S_x' = - P_B = - \frac{w x^2}{2L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

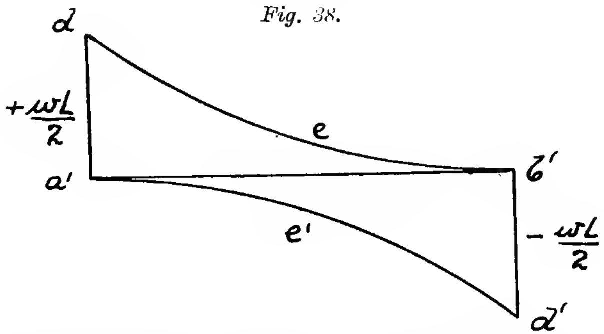
The curves of maximum positive and maximum negative shears are the parabolas $d e b'$ and $a'e'd'$, Fig. 38, of which (2) and (3) respectively are the

equations, and from this figure it is seen that, so far as magnitude alone is concerned, the shear at any point will be a maximum when the load extends up to the point from the farther support.

Case II.—Beam supported at the ends and acted upon by a uniformly-distributed dead load together with the uniform moving load of the above case.

Let w_1 be the moving load, and w_2 the dead load per unit of length. Then,

Fig. 38.



due to the dead load alone, the bending moment at any point D, distant x from A, will be (see Case VII, Chap. II):

$$M_x = \frac{w_2}{2}(Lx - x^2),$$

and the shear will be

$$S_x = w_2\left(\frac{L}{2} - x\right).$$

Hence, substituting w_1 for w in equations (1), (2) and (3) of the last case, for the combined dead and live (or moving) loads we shall have:

$$\text{maximum bending moment } M_x = \frac{w_1 + w_2}{2}(Lx - x^2) \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{,, \quad positive shear } \quad S_x = \frac{w_1 (L - x)^2}{2L} + w_2\left(\frac{L}{2} - x\right) \quad . \quad . \quad (5)$$

$$\text{and ,, \quad negative shear } \quad S_x' = -\frac{w_1 x^2}{2L} + w_2\left(\frac{L}{2} - x\right) \quad . \quad . \quad . \quad (6)$$

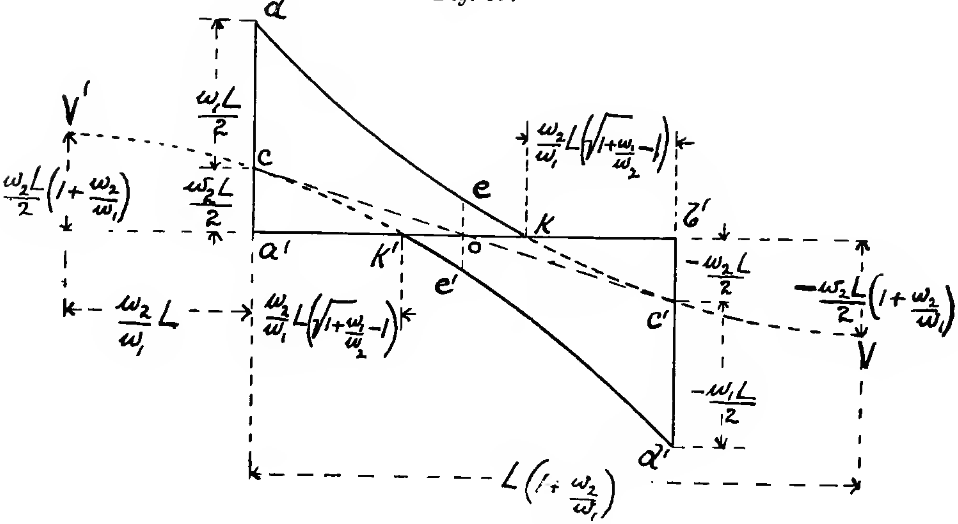
The diagram of maximum bending moments, therefore, is a parabola, similar to $a\,v\,b$, Fig. 37, of which (4) is the equation, and of which $v\,o$, the ordinate at

the centre, represents $\dfrac{(w_1 + w_2)\ L^2}{8}$. The diagrams of positive and negative shears are the parabolas d K V and d'K$'$V$'$, of which (5) and (6) respectively are the equations.

To find the point K$'$ put $S_x' = 0$ when

$$x^2 + 2\,L\,\frac{w_2}{w_1}\,x = \frac{w_2}{w_1}\,L^2 \quad \text{or} \quad x = \frac{w_2}{w_1}\,L\left(\sqrt{1 + \frac{w_1}{w_2}} - 1\right) = a'\,K'$$

Fig. 39.



Similarly, putting $S_x = 0$ in (5), we get:—

$$x = a'\,K = L - \frac{w_2}{w_1}\,L\left(\sqrt{1 + \frac{w_1}{w_2}} - 1\right), \text{ or } K\,b' = \frac{w_2}{w_1}\,L\left(\sqrt{1 + \frac{w_1}{w_2}} - 1\right).$$

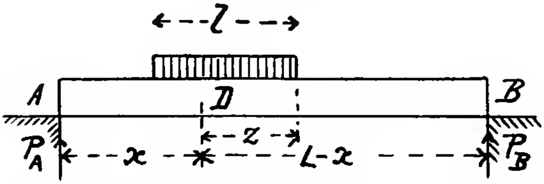
To find V$'$, differentiate equation (6) at equate to zero, when we get $-\dfrac{w_1\,x}{L} - w_2 = 0$, or $x = -\dfrac{w_2}{w_1}\,L$; and putting this value of x in (6) we get

$$S' = -\frac{w_1}{2\,L}\cdot\left(\frac{w_2}{w_1}\,L\right)^2 + w_2\left(\frac{L}{2} + \frac{w_2}{w_1}\,L\right) = +\,w_2\cdot\frac{L}{2}\left(1 + \frac{w_2}{w_1}\right).$$

In the same way, or from symmetry, we obtain, for the co-ordinates of V, $x = L\left(1 + \dfrac{w_2}{w_1}\right)$ and $S = -\,w_2\,\dfrac{L}{2}\left(1 + \dfrac{w_2}{w_1}\right)$. From Fig. 39 it is clear that

between a' and K' the shear is always positive, between K and b' it is always negative, and between K' and K it is sometimes positive and sometimes negative. Hence K' and K are the limiting points between which counter-bracing may be required in the case of braced girders.

Fig. 40.



Case III.—Beam supported at both ends and acted upon by a uniform moving load of length less than the span.

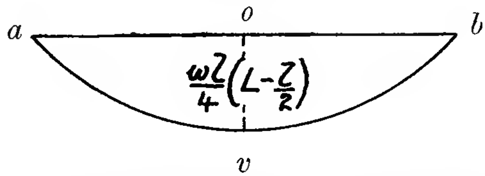
Let l be the length of the moving load, w the load per unit of length, x the distance from A of any point D of the beam, and z the distance of the right extremity of the load from D, Fig. 40.

Then :—

$$P_A = \frac{w.l}{L}\left(L - x - z + \frac{l}{2}\right) \text{ and the bending moment at D is}$$

$$M_x = \frac{w.l}{L}\left(L - x - z + \frac{l}{2}\right)x - w.\frac{(l-z)^2}{2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Fig. 41.



For a maximum at D, z must have such a value that $\frac{dM}{dz} = 0$;

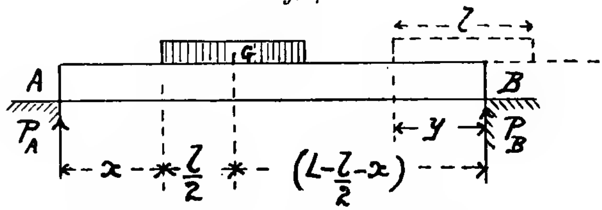
$$\text{or} \qquad -\frac{w\,l\,x}{L} + w\,(l - z) = 0$$

$$\text{or} \qquad z = l - \frac{l\,x}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

That is, for a maximum bending moment at any point D, $\frac{z}{l} = \frac{L - x}{L}$; or the

point D divides the length, l, of the load into two portions which bear the same ratio to one another as the corresponding portions, D B and D A, into which it divides the span.

Fig. 42.

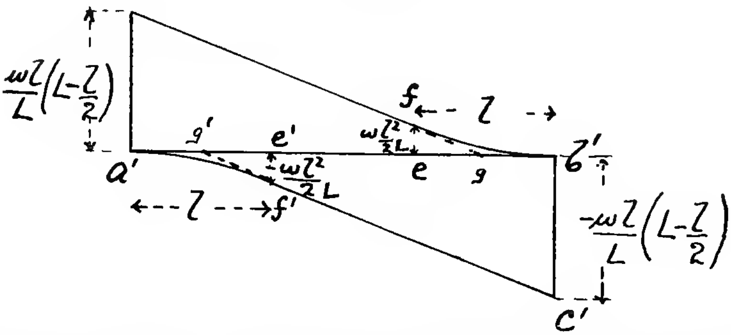


Substituting $l - \frac{l.x}{L}$ for z in (1) we get for the maximum bending moment at D,

$$M_{max} = \frac{w.l}{L}\left\{L - x - \frac{l}{2} + \frac{l.x}{L}\right\}x - \frac{w}{2}\cdot\frac{l^2x^2}{L^2}$$

$$= \frac{w.l.x}{L}(L - x)\left(1 - \frac{l}{2L}\right)$$

Fig. 43.



the equation to a parabola, $a\,v\,b$, Fig. 41, of which the ordinate $v\,o$ at the centre represents $w\,l\left(\frac{L}{4} - \frac{l}{8}\right)$, the maximum bending moment at the centre.

Maximum Shearing Force.—Assuming the load moving across the span from right to left, the maximum positive shear at D will occur when the front of the load is at the point, and the maximum negative shear when the rear of load is at D.

While a portion of length y of the load (Fig. 42) only is on the beam, the maximum shear under the front of the load will be $S = \frac{w\,y^2}{2\,L}$, or, for a point between b' and e, where $b'e = l$, the maximum positive shear will be given by the ordinate at the point to the parabola $b'f$, but when the load comes wholly on the beam the maximum positive shear at any point between e and A, distant x from A, will be:

$$S_x = P_A = w\,l.\frac{\left(L - x - \frac{l}{2}\right)}{L},$$

which is the equation to the straight line $c\,f\,g$, Fig. 43. The diagram of positive shears is, therefore, $a'c\,f\,b'$, and if we now assume the load to move across the span from left to right, we shall obtain a similar diagram $a'f'c'b'$ for the maximum negative shears.

Concentrated Loads.

Case IV.—Beam supported at both ends acted upon by a single load W rolling across.

Let W be at any point D, distant x from A, Fig. 44, then:—

$$P_A = W.\frac{L - x}{L} \quad P_B = W.\frac{x}{L}$$

and the binding moment at D will be:

$$M_x = P_A.x = P_B.(L - x).$$

If D now move towards B, P_A will decrease and the bending moment, $P_A.x$, will decrease at the same time; and, similarly, the bending moment at D will decrease if w move towards A. Hence, the maximum bending moment at any point, D, of the beam will occur when the load is at the point, and its value will be:

$$M_{max} = P_A.x = W.\frac{(L - x)}{L}.x \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The diagram of maximum bending moments, therefore, is a parabola $a\,v\,b$, Fig. 45, where $v\,o$, the ordinate at the centre, represents $\frac{W.L}{4}$.

Again, as the load W moves up from B to D, the shear $S_x\,(= P_A)$ at D will

increase from zero to $+ W.\frac{(L - x)}{L}$; on passing the point, however, and continuing to advance towards A, the shear will become negative and equal to $P_A - W = - P_B$, which diminishes, numerically, as W approaches A.

The positive shear, therefore, will be a maximum for any point when the

Fig. 44.

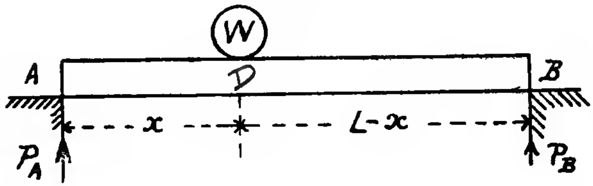


Fig. 45.

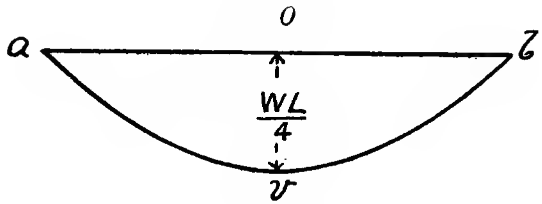
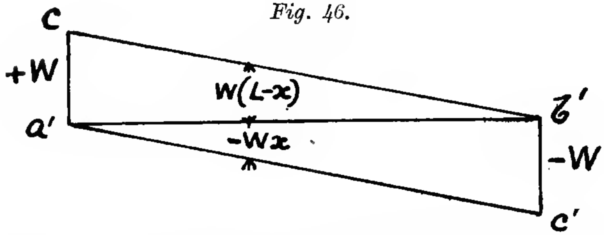


Fig. 46.



load is just to the right of the point; and the negative shear will be a maximum when the load is just to the left of the point, their values for a point distant x from A being, respectively,

$$S_x = W.\frac{L - x}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$S_x' = - \frac{W x}{L}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The diagram of positive shears, therefore, will be a triangle $a' c b'$, Fig. 46, and that of the negative shears a triangle $a' b' c'$.

Case V.—Beam supported at both ends and acted upon by two moving loads, W_1 and W_2, at a fixed distance apart.

Reasoning in the same way as before, it is easily shown that the maximum bending moment at any point will occur when either one or other of the loads is over the point; and the bending moment due to both W_1 and W_2 at a point under either load may be obtained by adding the bending moments at the point due to W_1 and W_2 found separately.

Let d be the fixed distance between W_1 and W_2, and suppose the loads to move across the span from right to left. Then, due to W_1 alone, the bending moment at a point distant x from A, when W_1 is over the point, will be as given by equation (1) of last case, and, therefore, will vary with x according to the ordinates of the parabola

$$_1M_x' = W_1 (L - x).\frac{x}{L} \text{ (4), or } a\, v_1\, b, \text{ Fig. 48.}$$

Now, W_2 will not come on the beam until W_1 has moved through a distance $b c = d$ from B, but as W_1 moves further to the left and comes to a point distant x from A, the bending moment at the point due to W_2 alone will be :—

$$_1M_x'' = W_2 (L - x - d) \frac{x}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the equation to the parabola $a\, v_2\, c$, Fig. 48.

Hence, for values of x between o and $L - d$, the bending moment under W_1, due to W_1 and W_2 together, will be :—

$$_1M_x = {_1M_x'} + {_1M_x''} = (W_1 + W_2)(L - x)\frac{x}{L} - W_2.d.\frac{x}{L} \quad . \quad . \quad (6)$$

the equation to a parabola $a\, V_1\, n$ which cuts the parabola $a\, v_1\, b$ at some point K.

The diagram of bending moments for *points under W_1*, therefore, will be $a\, V_1\, K\, b$, Fig 48.

In the same way, assuming the loads moving across the span from left to

right and x the variable distance of W_2 from A, we obtain, for *points under* W_2, the parabolas $a' v_2' b'$, $c' v_1' b'$ and n' K' $V_2 b'$, the equations to which are

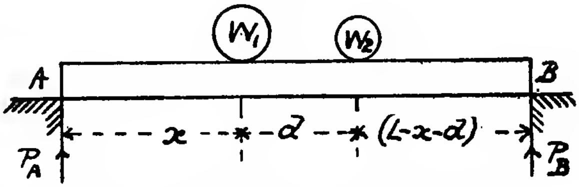
$$_2M_x' = W_2 . \frac{L - x}{L} . x. \quad (7) \quad \text{(due to } W_2 \text{ alone).}$$

$$_2M_x'' = W_1 \frac{(L - x + d)}{L} x - W_1 d. \quad (8) \quad \text{(due to } W_1 \text{ alone).}$$

and $$_2M_x = {_2M_x'} + {_2M_x''} = (W_1 + W_2)(L - x)\frac{x}{L} - W_1 \frac{L - x}{L} . d. \quad (9)$$

(due to W_1 and W_2 combined).

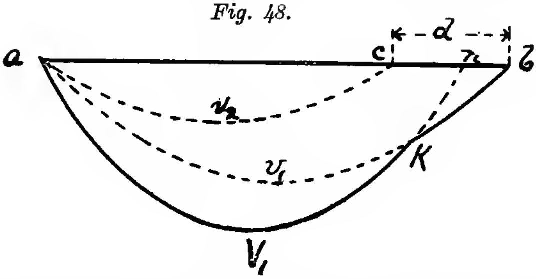
Fig. 47.



And the diagram of bending-moments will be a' K' $V_2 b'$.

Combining the diagram, a V_1 K b (Fig. 48), of maximum bending moments for points under W_1, with the diagram, a' K' $V_2 b'$ (Fig. 49), of maximum

Fig. 48.

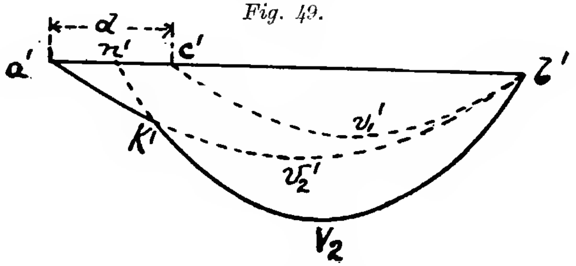


bending moments for points under W_2, we obtain the diagram, A'V_1 r' u B', Fig. 50; the ordinate to which, at any point, will give the actual maximum bending moment at the point.

From this diagram (Fig. 50) it is seen that for any point between A' and r the maximum bending moment will occur when W_1 is over the point, and for any point between r and B' the maximum bending moment will occur when W_2 is over the point.

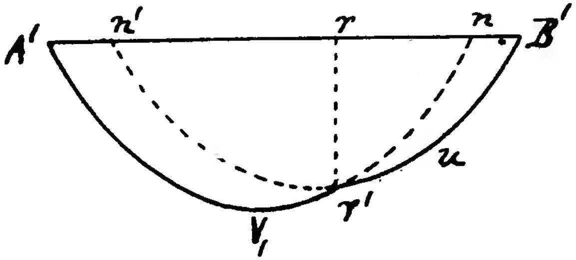
Again, from equations (6) and (9) it follows that, for any point distant x from A, W_1 at the point will give a greater or less bending moment than will W_2 according as ${}_1M_x - {}_2M_x = W_1 . \frac{L - x}{L} . d - \frac{W_2 . d . x}{L}$ is positive or negative; that is, according as x is less or greater than $\frac{W_1}{W_1 + W_2} . L \; (= A'r)$. (10)

Fig. 49.



This equation, however, will only give the limiting value of x, for which W_1 gives a greater bending moment than W_2, so long as the distance, d, between the loads is not greater than half the length of the span when the loads are equal,

Fig. 50.



or greater than some less distance, d', depending upon the ratio $W_1 : W_2$ when the loads are unequal. For if d be greater than $\frac{L}{2}$ (or d') the diagram of maximum bending moments will assume the form shown in Fig. 53, where u_2, and not r', gives the limiting value of x required.

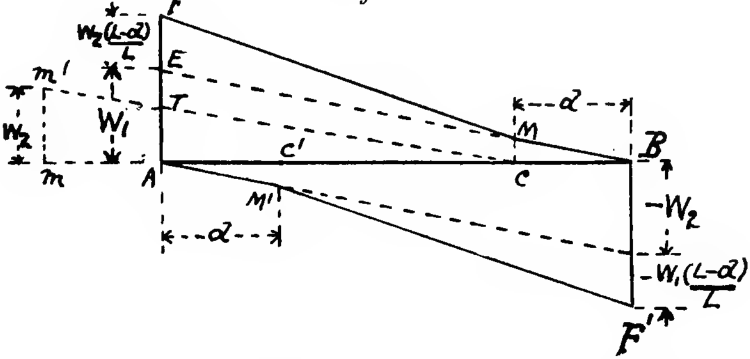
Point of Greatest Bending Moment.

From equation (6), by differentiation, we get:

$\frac{d\,M_x}{d\,x} = \frac{W_1 + W_2}{L}(L - 2x) - W_2 . \frac{d}{L}$, which vanishes when

$L - 2x = \frac{W_2}{W_1 + W_2} . d$, or $\frac{L}{2} - x = \frac{W_2}{W_1 + W_2} . \frac{d}{2} = AO - AE = EO$, Fig. 52, where E is the point of greatest maximum bending moment under W_1. Now, if G be the centre of gravity of W_1 and W_2, $GE = \frac{W_2}{W_1 + W_2} . d = 2 \times EO$.

Fig. 51.



The greatest maximum bending under W_1, therefore, will occur when W_1 and G, the centre of gravity of W_1 and W_2, are equidistant from, and on opposite sides of, the centre of the span. Similarly, by differentiating equation (9) and equating to zero, we shall find that the greatest bending moment under W_2 occurs when that load and G are equidistant from O.

Another Proof.—Equation (6) to the parabola $A' V_1 n$, Fig. 50, can be written

in the form $$- {}_1M_x = \frac{W_1 + W_2}{L}\left\{x^2 - \left(L - \frac{W_2 . d}{W_1 + W_2}\right)x\right\},$$

or $$\frac{W_1 + W_2}{L}\left(x - \left(\frac{L}{2} - \frac{W_2}{W_1 + W_2} . \frac{d}{2}\right)\right)^2 = \frac{W_1 + W_2}{L}\left(\frac{L}{2} - \frac{W_2}{W_1 + W_2} . \frac{d}{2}\right)^2 - {}_1M_x,$$

which is the equation to a parabola having its vertex at a point V_1 distant

$\frac{L}{2} - \frac{W_2}{W_1+W_2} \cdot \frac{d}{2}$ (11) from A, and at a distance $= \frac{W_1+W_2}{L}\left(\frac{L}{2} - \frac{W_2}{W_1+W_2} \cdot \frac{d}{2}\right)^2$ (12) below A′ B′. (12) gives the greatest bending moment under W_1, and (11) gives the point along the span at which it occurs—the same as before. Similarly, equation (9) can be written in the form:

$$\frac{W_1 + W_2}{L}(x - a)^2 = \frac{W_1 + W_2}{L} . a^2 - ({}_2M_x + W_1 d)$$

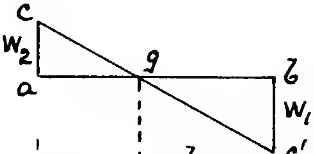


Fig. 52.

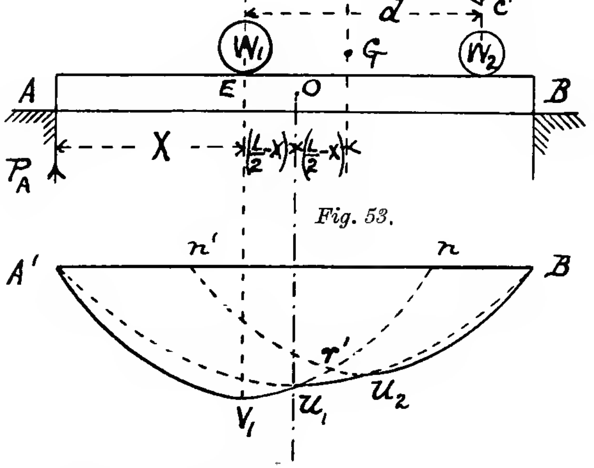


Fig. 53.

$\left(\text{where } a = \frac{L}{2} + \frac{W_1}{W_1 + W_2} \cdot \frac{d}{2}\right)$, giving for the vertex V_2 of the parabola $n' V_2 b'$, Fig. 49, the co-ordinates a and $\frac{W_1 + W_2}{L} . a^2 - W_1 . d$, and thus showing that the greatest bending under W_2 occurs when W_2 is at a distance to the right of O equal to $\left(\frac{g\,b}{2}, \text{Fig. } 52\right)$ half the distance of G from W_2, and that the value of this bending moment is $= \frac{W_1 + W_2}{L} a^2 - W_1 d$.

Diagram of Maximum Shears (Fig. 51).—The maximum positive shear at any point D, distant x from A, will occur when W_1 is just to the right of the point, and the maximum negative shear will occur when W_2 is just to the left of the point.

At a point under W_1, distant x from A, the shear due to W_1 alone will be

$$ {}_1S_x = W_1 \frac{L - x}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13) $$

Fig. 54.

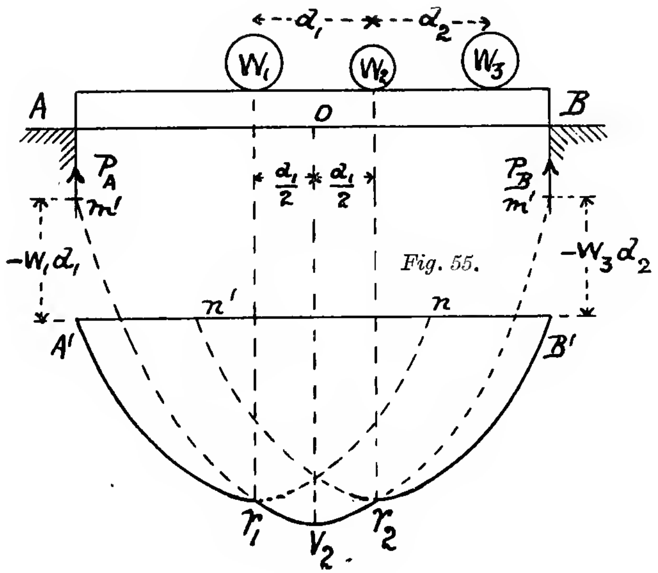


Fig. 55.

and that, at the same point, due to W_2 alone, will be

$$ {}_1S_x = W_2 \frac{(L - x - d)}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14) $$

provided x be not greater than $Ac = L - d$ (Fig. 51), for until W_1, moving towards the left, has come to c, where $Bc = d$, W_2 has not come on the span, and will, therefore, not produce any shear in the beam.

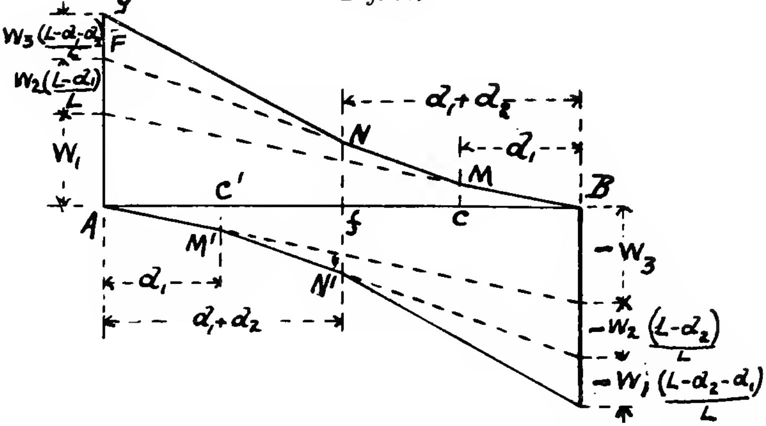
Now (13) is the equation of the straight BE where AE represents W_1, and

(14) that of the straight line C T, where $A\,T = W_2 . \frac{L - d}{L}$; therefore, if we make $E\,F = A\,T$ and join F to the intersection M of the vertical through C with B E, we obtain the diagram A F M B of maximum positive shears. The diagram A M′F′B of maximum negative shears may be obtained in the same way.

Case V.—Beam supported at both ends and acted upon by three moving loads at fixed distances apart.

The methods adopted in the last case for two moving loads can be extended to three or more loads. For example, if we have three loads, W_1, W_2 and W_3 at

Fig. 56.



distances of d_1 and d_2 apart (Fig. 54), the maximum bending moment for points under W_1 will be given by the equation:

$$ {}_1M_x = \{W_1 . (L - x) + W_2 (L - x - d_1) + W_3 (L - x - d_1 - d_2)\} \frac{x}{L} (= P_A . x) $$

$$ = (W_1 + W_2 + W_3)(L - x) . \frac{x}{L} - [(W_2 + W_3)\, d_1 + W_3 d_2] \frac{x}{L} \quad . \quad . \quad (1) $$

for points under W_2 it will be:

$$ {}_2M_x = \{W_1 (L - x + d_1) + W_2 (L - x) + W_3 (L - x - d_2)\} \frac{x}{L} - W_1 . d_1 \quad (2) $$

and for points under W_3 it will be:

$$ {}_3M_x = \{W_1 (L - x + d_1 + d_2) + W_2 (L - x + d_1) + W_3 (L - x)\} \frac{x}{L} $$
$$ - W_1 (d_1 + d_2) - W_2 . d_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3) $$

which are the equations to the parabolas $A' r_1 n$, $m' V_2 m'$ and $n' r_2 B'$ respectively. Fig. 55 (in this figure $d_1 = d_2$ and $W_1 = W_3$).

Now equation (1) may be written in the form

$$-{}_1M_x = \frac{W_1 + W_2 + W_3}{L}\left\{x^2 - \left(L - \frac{W_2 d_1 + W_3 (d_1 + d_2)}{W_1 + W_2 + W_3}\right)x\right\}$$

or
$$\frac{W_1 + W_2 + W_3}{L}(x - a_1)^2 = \left(\frac{W_1 + W_2 + W_3}{L}\right)a^2_1 - {}_1M_x \quad . \quad (4)$$

where
$$a_1 = \frac{L}{2} - \frac{W_2 d_1 + W_3 (d_1 + d_2)}{2(W_1 + W_2 + W_3)} = \frac{L}{2} - \frac{1}{2}\,G\,W_1,$$

where $G\,W_1$ is the distance of the centre of gravity of W_1, W_2 and W_3 from W_1.

Now (4) is the equation to a parabola $A'r_1n$, of which the co-ordinates of the vertex V_1 are

$$x = a_1 \text{ and } y \text{ (or } {}_1M_x) = \frac{W_1 + W_2 + W_3}{L} \,.\, a_1^2.$$

Hence the vertex V_1 of the parabola $A'r_1n$ (or the point under W_1, where the bending moment is greatest), and the centre of gravity of the total load are equidistant from the centre O of the span, and the maximum bending moment is ${}_1M_{max} = \dfrac{W_1 + W_2 + W_3}{L} \,.\, a_1^2$.

In the same way equations 2 and 3 can be written in the forms

$$\frac{W_1 + W_2 + W_3}{L}(x - a_2)^2 = \frac{W_1 + W_2 + W_3}{L} \,.\, a_2^2 - {}_2M_x - W_1 d_1 \quad . \quad (5)$$

and
$$\frac{W_1 + W_2 + W_3}{L}(x - a_3)^2 = \frac{W_1 + W_2 + W_3}{L} \,.\, a_3^2 - {}_2M_x$$
$$- W_1 (d_1 + d_2) - W_2 d_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Now (4), (5) and (6) show that the three parabolas $A'r_1n$, $m'V_2m'$ and $n'r_2B'$, Fig. 55, are portions of the same curve, but with different positions of the vertex. In fact, they are portions of the parabolic curve of bending moments due to a single rolling load equal to the sum of the several loads W_1, W_2 and W_3. They also show that the maximum bending moment under any load will occur when that load and the centre of gravity of the whole of the loads are equidistant from the centre of the span.

The above results are true for any number of moving loads so long as the loads are on the span when the position of their centre of gravity is taken, and, therefore, the curve of maximum bending moments will consist of a number of arcs of the same parabolic curve, the axis being always vertical, but the vertex occupying different positions for the different arcs. A diagram of bending-moments for four loads is shown in Fig. 73, p. 65.

Diagram of Maximum Shears, Fig. 56.—Proceeding as in the last case (Fig. 51), if we set off F g representing $W_3 \dfrac{(L - d_1 - d_2)}{L}$ and join g to the point of intersection, N, of F M with a vertical, f N, drawn at a distance $= d_1 + d_2$ from B, A g N M B will be the diagram of positive shears, and the diagram of negative shears can be similarly completed, as shown in Fig. 56.

Case VI.—Beam supported at both ends and acted upon by any number of moving loads at fixed distances apart.

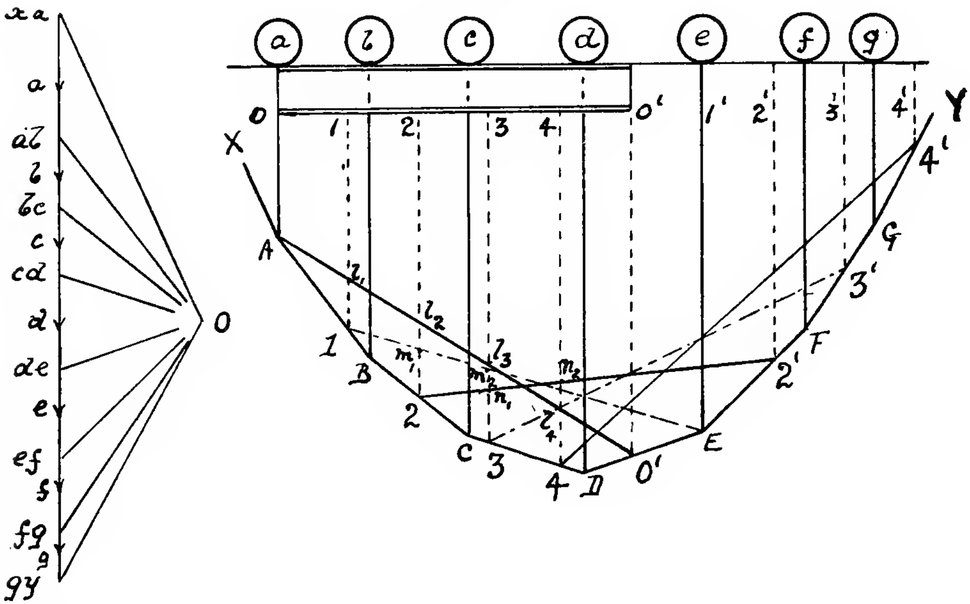
The methods adopted in the last case for obtaining the diagrams of maximum bending moments and maximum shearing forces for three loads can be applied to any number of loads, but the greater the number of loads the greater is the complication. Diagrams approximating closely enough for all practical purposes to the true forms can be readily obtained, however, as shown in the following paragraph.

Diagram of Maximum Bending Moments.—Let the loads be a, b, c, d, e, f, and g, and let their relative positions be as shown in Fig. 57. Along a vertical line lay off the distances $x\,a$–$a\,b$, $a\,b$–$b\,c$, $c\,d$–$d\,e$ . . . . $f\,g$–$g\,y$, representing the loads $a, b, c, \ldots g$ respectively; then, taking any pole O, and starting from any point x, draw X A, parallel to O–$x\,a$, to cut the vertical through load a in A; from A draw A B parallel to the radius vector O–$a\,b$ to cut the vertical drawn through load b in B and so on; thus obtaining the funicular polygon X A B C . . . G Y.

Now let the beam, in the first instance, be placed with its left terminal under load a, as shown in Fig. 57; then, as was explained in reference to Fig. 22, A B C D O′ will be a diagram of bending moments; and if the beam be divided into five equal parts (in practice the number should be at least ten) and through the points of division 1, 2, 3 and 4, vertical lines be drawn, the inter-

cepts $1l_1$, $2l_2$, $3l_3$ and $4l_4$ will give the bending moments at distances of one-fifth, two-fifths, three-fifths and four-fifths of the span respectively from the left terminal. Now, instead of moving the loads across the span from right to left, imagine the beam to move, under the loads, from left to right; then, when the beam has moved through a distance of one-fifth of the span—the left terminal coming to the point of division marked 1 and the right terminal to 1′—the new diagram of bending moments will be 1 B C D E, and the new bending

Fig. 57.

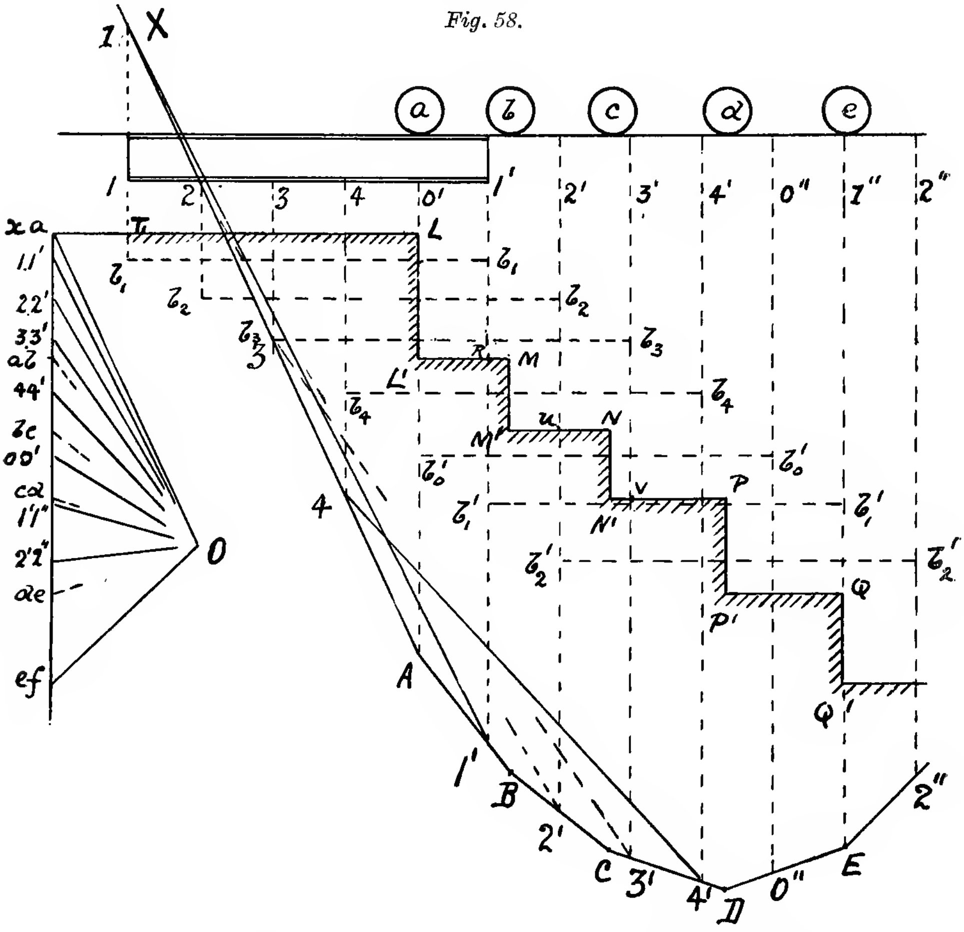


moments at the distances of one-fifth, two-fifths. . . . of the span will be given by the intercepts $2\,m_1$, $3\,m_2$. . . Giving the beam further successive shifts to the right through distances of one-fifth span, we obtain, in succession, the bending moment diagrams, 2 C D E 2′, 3 D E F 3′, 4 D E F G 4′. . . . .

The diagram of maximum bending moments, Fig. 59, can now be plotted as follows: Let $o\,o'$, divided into five equal parts, represent the span; then, from the points of division, erect ordinates 11′, 22′, 33′ and 44′, represent-

ing, respectively, the greatest of the bending moments obtained at the distances from the left terminal of the beam of one-fifth, two-fifths, three-fifths and four-fifths span; that is, make 1 1′ equal to the greatest of the distances

Fig. 58.



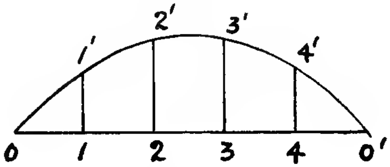
$1l_1$, $2m_1$, $3n_1$...(Fig. 57); 2 2′ equal to the greatest of the distances $2l_2$, $3m_2$, $4n_2$...(obtained either by scaling off or with a pair of dividers) and so on. A fair curve drawn through the points 0 1′ 2′ 3′ 4′ 0′, Fig. 59, will then

approximate more or less closely to the true form of the actual diagram of the maximum bending moments.

Diagram of Maximum Shearing Forces.—Draw the vector or force polygon and the funicular polygon as before, extending the line A X, however, as shown in Fig. 58, and adding the load-steps T_1 L L′ M M′ N . . . . as explained with reference to Fig. 24. When the right terminal of the beam is just to the right of load *a* the negative shear will be equal to L L′ or *a*, but the positive shear (and supporting force at left terminal) will be nil. Now suppose the girder to advance towards the right through a distance of one-fifth span, it will then occupy the position shown in the figure, and 1 A 1′ will be the corresponding diagram of bending moments.

Now draw from the pole O a line parallel to the closing line 1′1 to cut the

Fig. 59.

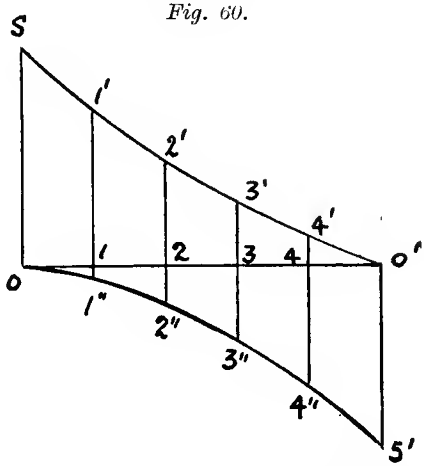


line of loads in 1 1′ and through this latter point a horizontal line $b_1 b_1$ under the girder; $b_1 b_1$ will then be the base line and b_1 T_1 L L′ R b_1 the diagram of shearing forces for the girder in the position shown, as was explained with reference to Figs. 22, 23 and 24. Advancing the girder in this way through successive intervals of one-fifth (or one-tenth) span, we obtain, in succession, the diagrams of bending moments 2 A B 2′, 3 A B C 3′ . . . . , and drawing from pole O lines parallel to the closing lines 2′2, 3′3, etc., to cut the line of loads in the points 2 2′, 3 3′, etc., and through these points horizontal lines under the girder, we obtain the respective base lines $b_2 b_2$, $b_3 b_3$, etc. The vertical (dotted) lines drawn at intervals of one-fifth span will divide each base line into five equal parts, and the ordinate to the stepped figure at each point of division will give the shearing force corresponding to that point, the shearing force being positive or negative according as the ordinate is measured upwards or downwards from the point.

To plot the diagram of maximum shearing forces (Fig. 60) take a pair of

dividers and obtain, first, the greatest of the ordinates measured downwards to the stepped figure for the right extremities b_1, b_2, b_3, etc., of the base lines and lay it off along O′S′. Then do the same for points at one-fifth span from the right extremities of the base lines, and lay it off along 4 4″, and so on. Now obtain for points at one-fifth span from the right extremities of the base lines the greatest of the (positive) ordinates measured upwards and lay it off along 4 4′. Repeating this operation (which is much more rapidly done than would appear

Fig. 60.



from this description) the diagram shown in Fig. 60 can be completed. It approximates more or less closely to the true form of the diagram of maximum positive and maximum negative shears. A method of determining the greatest maximum bending moment and the point along the span at which it occurs will be explained in the next chapter, together with a method of determining the greatest maximum shearing force.

CHAPTER IV.

Graphical Determination of the Maximum Bending Moment and Maximum Shear due to a Train-Load.

As a preliminary to the design of railway underbridges it is now customary to prepare tables of "equivalent uniform live loads" for spans varying from 5 feet to 200 feet. The train-load usually taken for any particular railway and span as a basis of calculations is the locomotive, or, for larger spans, the combination of locomotives in use on the line, which, coupled together in the form of a continuous train, gives either:

1. The maximum bending moment, or
2. The maximum shear.

If M be the maximum bending moment, and S the maximum shear produced by the axle-loads as the "type-train" crosses the bridge, and if l be the effective span, then, for each track, the "equivalent uniform load derived from the maximum bending moment" will be: $W_1 = \frac{M}{\frac{l}{8}}$, and that "derived from the maximum shear" will be: $W_2 = 2\,S$. In the tables of equivalent uniform live loads, however, these values are usually increased by $2\frac{1}{2}$ per cent. to provide for possible future increase of the axle-loads of the type-trains.

Effective Span.—To determine the effective span, let the clear span (Fig. 61), be $2\,c$, the total length of each of the girders $2\,(c + a)$, and the uniform load per unit length w. The bending moment diagram will then consist of the three parabolas, $d\,e$, $e\,f\,g$, and $g\,h$ (Fig. 62), and if the parabola $e\,f\,g$ be continued, it will cut the base-line $d\,h$ in k and l, where the distance $k\,l$ represents, so far as the bending stresses are concerned, the effective span of the girders. The

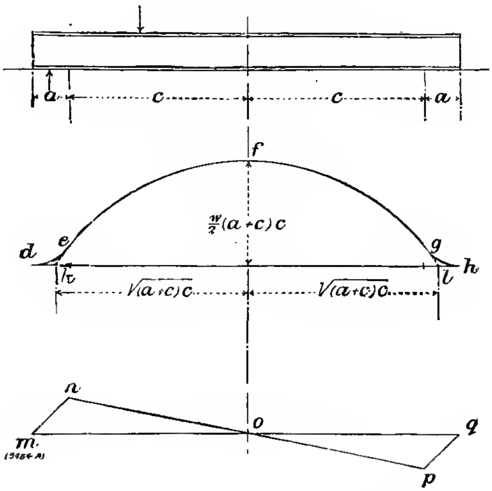
effective span $k\,l$ may also be determined as follows:—Each of the abutment reactions will be $= w\,(c + a)$, and will act at a distance $= c + \frac{a}{2}$ from the centre of the span; also, the load on each half of the girder will be $= w\,(c + a)$, and the distance of its centre of gravity from the centre of span will be $= \frac{1}{2}\,(c + a)$. Hence the bending moment at the centre will be

$$= w\,(c + a)\left(c + \frac{a}{2}\right) - w\,(c + a)\,\frac{(c + a)}{2} = w\,(c + a)\,\frac{c}{2} = w\,\frac{c'^2}{2},$$

where $2\,c'$ is the effective span $k\,l$.

Therefore the effective span $= k\,l = 2\,\sqrt{(c + a)\,c}$.

Figs. 61, 62 and 63.



So far as the shearing stresses are concerned, it is clear from the shearing force diagram, Fig. 63, that the effective span is the same as the clear span.

Maximum Bending Moment.—For a given type-train and a girder-bridge of given span, the greatest bending moment occurs—at a point near the centre—when the train extends over the whole length of the bridge, and when at the same time the heaviest wheels of an engine are near the centre of the span.

In most cases we are thus enabled to see at a glance what wheel-loads on the bridge will give the greatest bending moment; while in the others it is only necessary to test two, or, at most, three, sets of wheel-loads to obtain the one required. For example, if we take a piece of paper, or a scale, on the edge of which a distance of 50 feet is marked, and move it under the wheel diagram in Fig. 65 we see at once that the axle-loads which will be on a bridge of this span when maximum bending occurs are those marked *a, b, c, d, e, f* and *g*. If, however, we proceed in the same way for a span of 100 feet, it will appear doubtful as to whether the axle-loads marked f'' to a', or those marked f'' to b', give the greatest bending moment. In this case, therefore, both these load systems will require to be tried, when it will be found that the first is the one which gives the greatest bending moment.

Point of Greatest Bending Moment.—When a series of wheel-loads move over a beam, the bending moment at any point along the span will obviously be a maximum when the point is under one or other of the loads. The position of any load D, for which the bending moment under it will be a maximum, may be readily determined as follows.

In Fig. 64 let R be the total load upon the beam and R′ the load upon the portion A D, and let the distances of their centres of gravity from D be a and a' respectively. Then the bending moment under D will be

$$M = P_A(x + a) - R'a'$$

$$= R\frac{2c - x}{2c}(x + a) - R'a'$$

Therefore $$\frac{dM}{dx} = \frac{R}{2c}(2c - 2x - a)$$

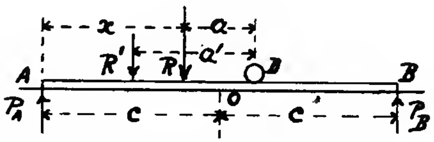
which vanishes when $2c = 2x + a$, or $c - x = x + a - c = \frac{a}{2}$. That is, "the maximum bending moment under any load D occurs when that load and the centre of gravity of the whole series on the span are equidistant from the centre of the span."

By the application of this rule the maximum moment under any wheel can easily be determined, but in the present case it is only the greatest of these

maxima that is required. Now, as the greatest bending moment produced by a train-load as it moves over a bridge occurs at or near the centre of the span, and when the heaviest wheels of an engine are in the neighbourhood of that point, it is only necessary to determine the points at which the maximum moments occur for two, or, at most, three of those wheels in order to obtain the point required. Further, if of two wheels A and B the maximum for A occurs nearer the centre than does that for B, A will generally be the one which gives the greater maximum bending moment.

Description of the Diagram.—The method found most convenient by the writer for determining on one diagram the greatest bending moments and shears

Fig. 64.



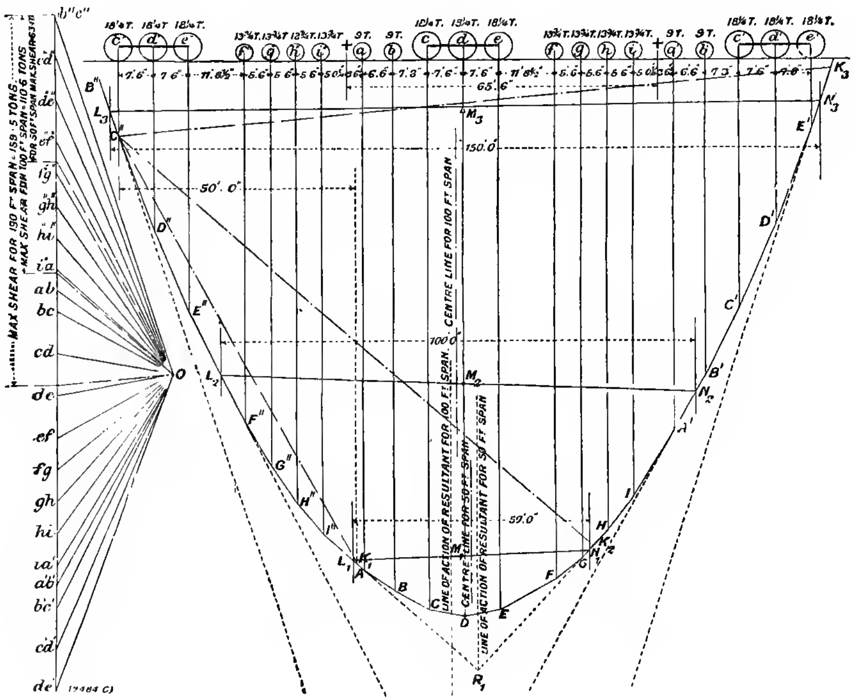
for a wide range of spans is illustrated in Fig. 65 by reference to a train of six-wheels coupled engines. The wheel diagram is laid off to a scale of $\frac{1}{8}$ inch to a foot, and the axle-loads are laid off along the line $b'' c''$–$d' e'$, to a scale of $\frac{1}{16}$ inch to a ton; $b'' c''$–$c'' d''$ representing load c''; $c'' d''$–$d'' e''$ representing d'', and so on.

Opposite the mid-point of $c\,d$–$d\,e$ (representing load d), the pole O is taken at a distance from it representing 50 tons on the load scale, and the funicular polygon B″ C″ D″. . . K_3 drawn. The funicular polygon thus obtained is suitable for all spans between 40 feet and 150 feet in length. For spans less than about 40 feet the pole O should be taken nearer to the line of loads, and for spans exceeding about 150 feet it should be taken further away.

Application of the Diagram to the Determination of Greatest Bending Moments.—For a span of 50 feet it has already been noticed that the loads on the bridge, when the greatest bending moment occurs, will be those marked a to g. Hence I″ A will be the first, and G H the last link of the corresponding funicular polygon; and as these intersect in R_1, the centre of gravity R of the loads a to g will lie vertically above R_1 (see note on page 13). From the rule already proved,

we know that if the greatest bending moment occurs under load d, the centre, o', of the span must lie midway between d and R; and if it occurs under load e, then o' will be midway between e and R. As the centre of gravity R, however, lies much nearer to d than e, d will be the load under which the bending moment will be greatest. Bisect, therefore, the distance between the verticals through

Fig. 65.



R_1 and d, and obtain the centre-line for a 50-foot span; then lay off, horizontally, a distance representing 25 feet on either side of it, project vertically downwards on to the funicular polygon at L_1 and N_1, and join these points. L_1 A B C . . . G N_1 will then be a bending moment diagram for the particular loads and span given, and D M_1 will represent the greatest bending moment required. If D M_1 be

measured in feet on the scale of lengths, and the distance of O from the line of loads be such as to represent 50 tons on the load scale, the bending moment required will be $M = D M_1 \times 50$ tons-feet.

On the diagram $D M_1$ was found to measure a distance representing, on the scale of lengths, $13 \cdot 125$ feet, so that the greatest bending moment, $M = 13 \cdot 125 \times 50 = 656 \cdot 25$ tons-feet, and the equivalent uniform load, $W = M \times \frac{8}{50} = 105$ tons.

Proceeding in the same way as before, the greatest bending moment for a span of 100 feet is represented by $D M_2$, and for a span of 150 feet by $D M_3$. $D M_2$, on the scale of lengths, represents $50 \cdot 2$ feet, giving as the greatest bending moment M, $50 \cdot 2 \times 50 = 2{,}510$ tons-feet; and as the equivalent uniform load $W = \frac{2{,}510 \times 8}{100} = 200 \cdot 8$ tons. The corresponding values for the 150-foot span are $110 \cdot 375$ feet, $5518 \cdot 75$ tons-feet, and $294 \cdot 33$ tons respectively.

Note.—After moving the scale, or strip of paper on which the terminals and centre of the span are marked, under the loads, and thus obtaining what appears to be the load system giving the greatest maximum bending moment, and after determining the position of the centre of gravity of this load system, it must be noted whether this load system still remains on the span or not when the girder is placed with its centre midway between the centre of gravity and the nearest wheel load. Occasionally it will be found that when the span is shifted into the latter position, of the load system first selected a load near one end will now be off the span and probably a new load will have come on at the other end. This new load system should then be taken, and the position of its centre of gravity and the new position of the span determined.

A very little practice, however, will enable a student to ascertain, with a rapidity and certainty unequalled by any other method, the greatest maximum bending moments and the points at which they occur for a wide range of spans, and for any train load.

Maximum Shear.—The greatest shear produced in a beam by a series of wheel-loads moving over it will always occur at an abutment, and when one or other of the loads is just on the point of moving off the beam.

In Fig. 66 loads W_1, W_2 . . . . . W_{n-1} are represented as acting upon a beam

A B of span L; the next load, W_n, being off the beam and at a distance z to the right of B. Let W_1, at first, act just to the right of A, and let the resultant, R, of the remaining loads W_2, $W_3 \ldots\ldots W_{n-1}$ on the beam act at a distance x from B; the shear at A will then be $S_1 = W_1 + R\frac{x}{L}$. Now suppose the loads to move towards the left through a distance a_1, so that W_2 comes to A, R to a point distant $x + a_1$ from B, and (assuming a_1 to be greater than z) W_n to a point on the beam distant $a_1 - z$ from B; the shear at A will then be:

$$S_2 = R\frac{x + a_1}{L} + W_n\frac{a_1 - z}{L},$$

the increase being

$$S_2 - S_1 = \frac{Ra_1}{L} + W_n\frac{a_1 - z}{L} - W_1 \quad \ldots\ldots \quad (1)$$

According, therefore, as $\frac{Ra_1}{L} + W_n\frac{a_1 - z}{L}$ is greater or less than W_1 will the loads W_2 or W_1 give the greater shear.

If z were greater than a_1, the term $W_n\frac{a_1 - z}{L}$ would not come in the expression (1) at all. Hence the limiting values of $S_2 - S_1$ are

$$\frac{Ra_1}{L} - W_1 \text{ and } (R + W_n)\frac{a_1}{L} - W_1.$$

If $\frac{R}{L}$ be greater than $\frac{W_1}{a_1}$, W_2 will give the greater shear; but if $\frac{R}{L}$ be less, and $\frac{R + W_n}{L}$ greater than $\frac{W_1}{a_1}$ equation (1) should be referred to.

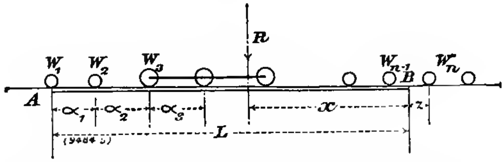
In practice the greatest shear will occur, usually, when the first of the heaviest wheels of an engine is at an abutment. For examples:—Referring to the train of six-wheels coupled engines shown on the diagram, Fig. 65, if wheel a were just moving off a span of 50 feet to the left, the other wheels on the span would be b, c, d, e, f, and g, and R would be $= 91\frac{1}{4}$ tons, W_1 (or a) 9 tons, and a_1 6·5 feet.

Therefore, since $\frac{R}{L} = \frac{91 \cdot 25}{50}$ is greater than $\frac{W_1}{6 \cdot 5} = \frac{9}{6 \cdot 5}$, b at the left abutment

would give a greater shear than would a. If b were now moved up to the left abutment, h would come on the span, R would become = 96 tons, and $a_1 = 7' 3''$.

Therefore, since $\frac{R}{50} = \frac{96}{50}$ is greater than $\frac{b}{a_1} = \frac{9}{7 \cdot 25}$ wheel c at the left abutment would give greater shear than would b. Now move c up to the abutment, then i will come on the span. R (now the resultant of the loads d, e, f, g, h, and i)

Fig. 66.



will be reduced to $91 \cdot 5$ tons, and a_1 increased to $7 \cdot 5$ feet, and since $\frac{R}{L} = \frac{91 \cdot 5}{50}$ is now less than $\frac{c}{7 \cdot 5} = \frac{18 \cdot 25}{7 \cdot 5}$, and also $\frac{R + W_n (\text{or } a^1)}{L} = \frac{91 \cdot 5 + 9}{50}$ less than $\frac{c}{7 \cdot 5} = \frac{18 \cdot 25}{7 \cdot 5}$, c at the left abutment would give a greater shear than would d.

In the same way it can easily be shown that for all spans the abutment reaction (and therefore the shear) will be greatest when the first of the heaviest wheels is just arriving at the abutment.

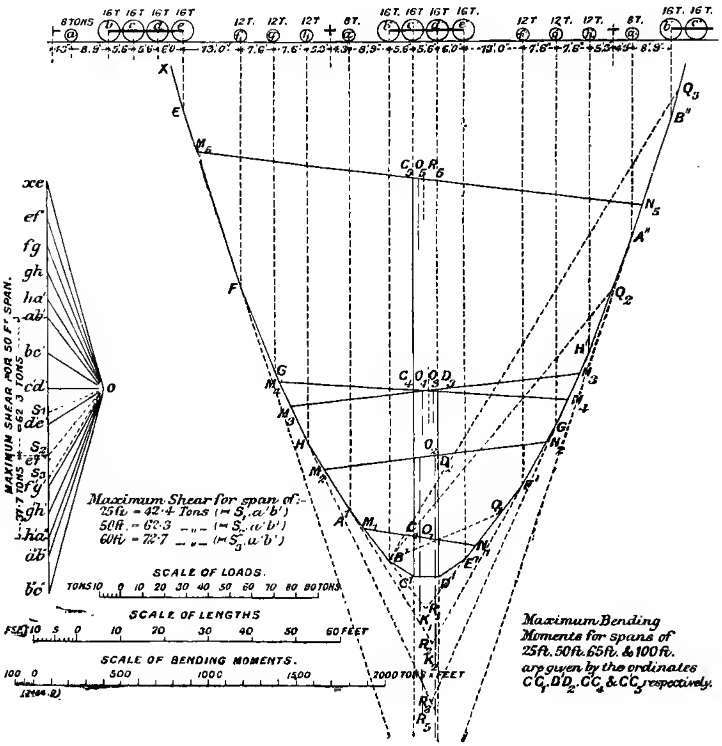
Graphical Determination of the Greatest Shear due to a Train-Load.—The method devised by the writer to render this operation as simple as possible is illustrated on the diagram, Fig. 65. Through the centre of the controlling wheel c'' (the first of the heaviest wheels in this case) a vertical line is drawn to cut the funicular polygon in C'', and at a distance from it equal to the span another vertical is drawn to cut the funicular polygon in a point K_1, say. Joining K_1 and C'' we obtain $K_1 C'' D'' E'' \ldots K_1$, the (Culman) diagram of bending moments for the span when the controlling load is at an abutment. Drawing from O, in the polar diagram, a line parallel to $K_1 C''$, this line will cut the line of loads in a point, the distance of which, from $b'' c''$, will represent the abutment reaction or the greatest shear required.

For a span of 50 feet the maximum shear, as noted on the diagram, is $63 \cdot 1$ tons.

The maximum shear for a 100-foot span (obtained by drawing a line from O parallel to $K_2 C''$) is 110·6 tons, and that for a span of 150 feet (obtained by drawing from O a line parallel to $K_3 c''$) is 159·5 tons.

In the diagram, Fig. 65, only the left abutment reactions are shown, but in

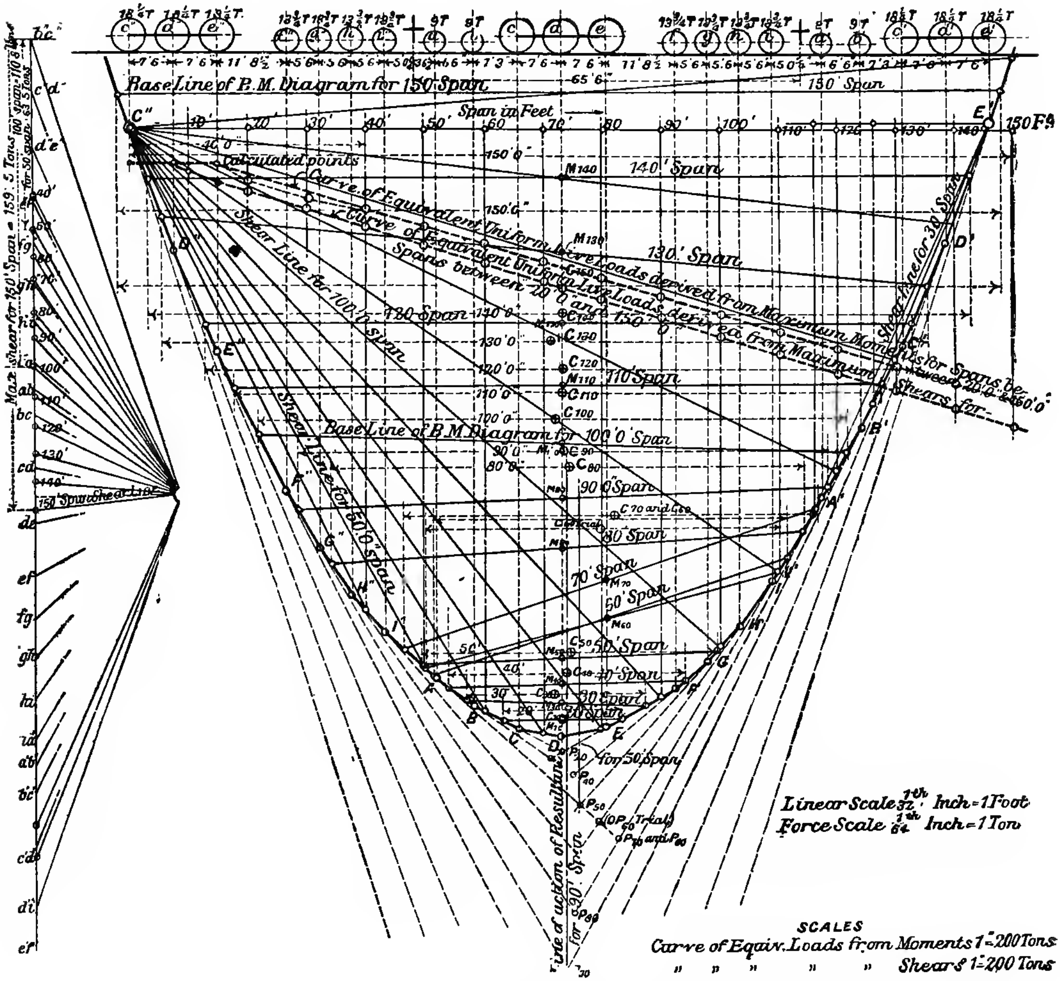
Fig. 67.



practice it may be necessary to determine the right abutment reactions as well. The controlling load for these latter reactions is e', or e_1, but as the reactions themselves were found to be less, for the spans given, than the corresponding reactions at the left abutment, their values were not noted on the diagram. For spans less than about 37 feet, however, the greatest shear will occur at the right

abutment when the wheel *e* is over the point. To obtain the maximum shear, therefore, for a span less than 37 feet we must proceed as follows:—To the left

Fig. 68.



of E, and at a distance from it representing the span, draw a vertical line to cut the funicular polygon in a point S, say, and from O, in the force-diagram, draw

a line parallel to E S to cut the line of loads in a point T, say, then T − ef will represent the shear required.

In the diagram, Fig. 67, the method of determining the maximum bending moment and shear is illustrated by reference to a train of eight-wheels coupled engines.

In Fig. 68 the complete graphical solution is given for the train of six-wheels coupled engines above referred to; the greatest maximum bending moments

Span.	Equivalent Uniform Live Loads, in Tons, Derived from—		Distance of Centre of Span, for Maximum Bending, to Left of Wheel *d* or *e*.
	Maximum Bending Moments.	Maximum Shears.	
Feet. 20	54·75	68·44	Feet. 0·00
30	73·91	87·52	+ 1·04
40	88·23	105·04	− 0·85
50	105·02	126·16	− 1·39
60	124·53	144·72	− 0·90 (*e*)
70	143·22	162·78	− 0·90 (*e*)
80	160·49	182·26	− 0·94
90	179·90	202·24	− 0·36
100	199·95	221·37	+ 1·32
110	217·92	241·44	+ 0·12
120	235·36	261·34	+ 0·10
130	254·25	280·02	+ 2·44
140	274·25	298·96	+ 0·09
150	294·38	318·92	+ 0·08

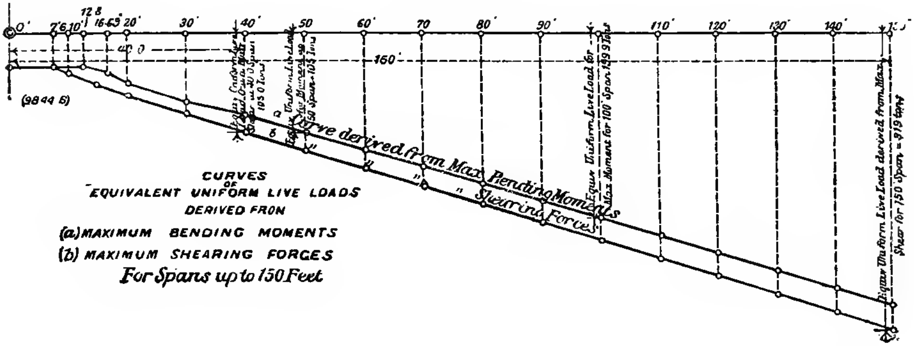
and greatest maximum shears being obtained for all spans, at intervals of 10 feet, from 20 feet to 150 feet.

Scaling off the maximum bending moments and maximum shears from this diagram and doubling each of the latter, and dividing each of the former by one-eighth of the corresponding span $\left(W = M \div \frac{l}{8}\right)$ the equivalent uniform live loads given in the table above are obtained. In this table the distances from wheel *d*, or *e*, at which the greatest maximum bending moments occur are

also entered, being regarded as positive when measured to the left from d, or e, and negative when measured to the right.

Taking the spans given in the above table as abscissæ and the equivalent uniform live loads derived from maximum bending moments as ordinates, the

Fig. 69.



curve of equivalent uniform live loads derived from maximum bending moments shown in Fig. 69 is obtained. In a similar way the curve of equivalent uniform live loads derived from maximum shears is obtained. From this diagram the equivalent uniform live load for any span can be readily obtained.

CHAPTER V.

ANALYTICAL METHODS OF DETERMINING THE MAXIMUM BENDING MOMENT AND MAXIMUM SHEAR DUE TO A TRAIN-LOAD.

IN preparing tables of equivalent uniform live loads for a given type-train, it is necessary to determine, for each span, the maximum bending moment and maximum shear produced as the train moves across the span. These determinations may be effected analytically, graphically, or by a combination of the two methods; whatever method be adopted, however, the work involved will be much facilitated by observing at the outset that:

1. The maximum bending moment will occur under one of the heaviest wheels of an engine when that wheel is at, or in the neighbourhood of, the centre of the span, and when, at the same time, the train-load extends over the whole span.

2. The point of maximum bending moment and the centre of gravity of the whole oad on the span when maximum bending occurs are equidistant from the centre of the span.

3. The maximum shear will always occur at an abutment, and when the span is fully loaded; and, usually, when the first of the more heavily loaded wheels of an engine is on the point of moving off the span.

Analytical Method.—Let W_1 be the heaviest of the wheel-loads of an engine, and L the span; then, for all spans from zero up to some particular value of $L = L_1$, say, W_1 at the centre will give the maximum bending moment, $\frac{W_1 L}{4}$, and the corresponding equivalent uniform live load will be $2 W_1$.

For two loads, W_1 and W_2 on the span, the maximum bending moment will occur under the heavier load, W_1, when this load and G, the centre of gravity of

both, are equidistant from O, the centre of the span. Let A B, Fig. 70, represent the span, and let d_1 be the (fixed) distance between the loads; then, for maximum bending,

$$GO = Om = \frac{Gm}{2} = \frac{Gn}{2}\frac{W_2}{W_1} = \frac{mn}{2}\frac{W_2}{W_1+W_2} = \frac{W_2}{2(W_1+W_2)}.d_1.$$

Reaction at

$$B = P_B = (W_1 + W_2)\frac{GA}{AB} = (W_1+W_2)\frac{\left(\frac{L}{2} - GO\right)}{L},$$

and maximum bending moment $= P_B.Bm$

$$= \frac{W_1+W_2}{L}.\left(\frac{L}{2} - GO\right)\left(\frac{L}{2} - Om\right)$$

$$= \frac{W_1+W_2}{L}\left(\frac{L}{2} - \frac{W_2}{2(W_1+W_2)}.d_1\right)^2$$

$$= \frac{((W_1+W_2)L - W_2.d_1)^2}{4.L.(W_1+W_2)} \quad \ldots\ldots \quad (1)$$

Fig. 70.

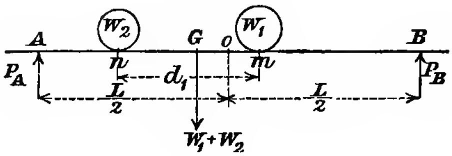
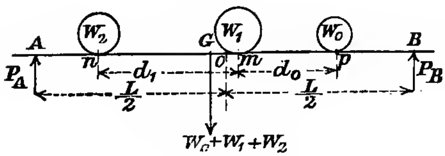


Fig. 71.



The maximum bending due to the two loads W_1 and W_2, therefore, will be greater or less than that produced by the single load W_1, according as the expression (1) is greater or less than $W_1 \frac{L}{4}$.

That is, according as $((W_1 + W_2) L - W_2.d_1)^2 - W_1 (W_1 + W_2) L^2$ $= W_2 (W_1 + W_2) \left\{L^2 - 2 L.d_1 + \frac{W_2}{W_1 + W_2}.d_1^2\right\}$ is positive or negative.

Hence, when the span, L, is greater than $d_1 \left(1 + \sqrt{\frac{W_1}{W_1 + W_2}}\right)$. (2)

the two loads $W_1 + W_2$ will give a greater maximum bending moment than will the single load W_1. Again, if three loads, W_0, W_1 and W_2 (Fig. 71), be on the span, the distance of their centre of gravity, G, to the left of O (the centre of span), when maximum bending occurs, will be:

$$G O = O m = \frac{G m}{2} = \frac{W_2.n m - W_0.p m}{2 (W_1 + W_2 + W_0)} = \frac{W_2.d_1 - W_0.d_0}{2 (W_1 + W_2 + W_0)};$$

the reaction at the left terminal A will be:

$$P_A = (W_1 + W_2 + W_0) \frac{G B}{A B} = (W_1 + W_2 + W_0) \frac{\left(\frac{L}{2} + G O\right)}{L},$$

and the maximum bending moment, which occurs under wheel W_1, will be

$$P_A.A m - W_2.n m$$

$$= (W_1 + W_2 + W_0) \frac{\left(\frac{L}{2} + G O\right)}{L} \left(\frac{L}{2} + o m\right) - W_2.d_1$$

$$= \frac{(W_1 + W_2 + W_0)}{L} \left(\frac{L}{2} + \frac{W_2.d_1 - W_0.d_0}{2 (W_1 + W_2 + W_0)}\right)^2 - W_2.d_1$$

$$= \frac{((W_1 + W_2 + W_0) L + W_2.d_1 - W_0.d_0)^2}{4.L.(W_1 + W_2 + W_0)} - W_2.d_1 \quad . \quad (3)$$

Hence three loads will give a greater or less maximum bending moment than will two according as (3) is greater or less than (1); that is, according as the expression

$$\left\{(W_1 + W_2 + W_0) \frac{L}{4} + \frac{W_2 d_1 - W_0 d_0}{2} + \frac{(W_2 d_1 - W_0 d_0)^2}{4.L.(W_1 + W_2 + W_0)} - W_2.d_1\right\}$$
$$- \left\{(W_1 + W_2) \frac{L}{4} - \frac{W_2 d_1}{2} + \frac{W_2^2 d_1^2}{4.L.(W_1 + W_2)}\right\},$$

or

$$L^2 - 2 L.d_0 - \left\{\frac{W_2^2 d_1^2 + 2 W_2 (W_1 + W_2) d_1 d_0 - W_0 (W_1 + W_2) d_0^2}{(W_1 + W_2)(W_1 + W_2 + W_0)}\right\}$$

is positive or negative.

Hence three loads will give a greater or less maximum bending moment according as

$$L \text{ is} > \text{or} < \left\{ d_0 + \frac{(W_1 + W_2)\, d_0 + W_2\, d_1}{\sqrt{(W_1 + W_2)\,(W_1 + W_2 + W_0)}} \right\}. \quad . \quad (4)$$

Taking, for example, the six-wheels coupled engine shown in Fig. 72, $W_1 = W_2 = W_0 = 18\frac{1}{4}$ tons, and $d_1 = d_0 = 7 \cdot 5$ feet. From equation (2), we find that one wheel will give a greater maximum bending moment than two for spans up to $7 \cdot 5\,(1 + \sqrt{\tfrac{1}{2}}) = 12 \cdot 8$ feet, and two will give a greater maximum than will three or one for spans ranging between $12 \cdot 8$ feet, and

$$7 \cdot 5 + \frac{(3 \times 7 \cdot 5)}{\sqrt{6}} = 16 \cdot 69 \text{ feet.}$$

For spans up to $12 \cdot 8$ feet, equivalent uniform load $= 2 . W_1 = 36 \cdot 5$ tons.

From (1), when $L = 16 \cdot 69$ feet, maximum bending moment

$$= \frac{((18 \cdot 25 + 18 \cdot 25)\, 16 \cdot 69 - 18 \cdot 25 \times 7 \cdot 5)^2}{4 \times 16 \cdot 69\,(18 \cdot 25 + 18 \cdot 25)} = 92 \cdot 64 \text{ foot-tons,}$$

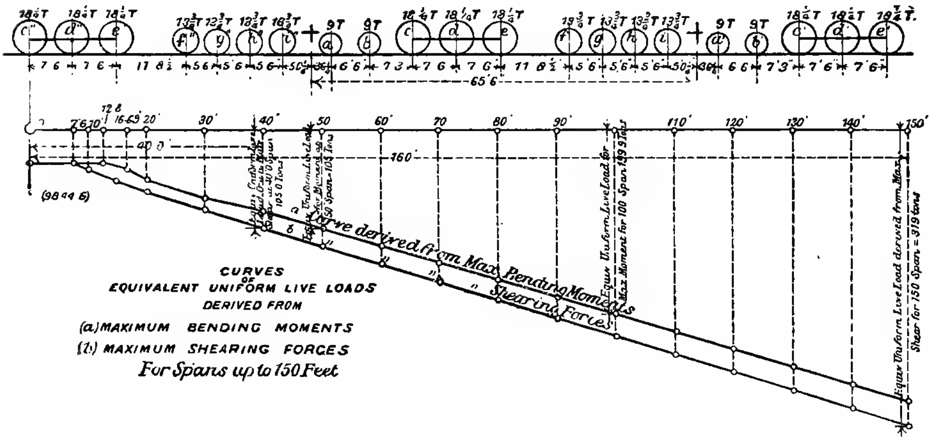
and equivalent uniform load $= 92 \cdot 64 \times \dfrac{8}{16 \cdot 69} = 44 \cdot 4$ tons.

Proceeding as above, the limits of the spans between which four, five, or any given number of loads will give the greatest maximum bending moments can be found; but this mode of procedure is not to be recommended, excepting in the case of very short spans, or when absolute accuracy is required. In practice, for any given span, the load system giving the greatest maximum bending moment may be determined very readily as follows:—Lay off the wheel-diagram of the type-train on any suitable scale, as shown in the diagram, Fig. 72, and underneath it move a strip of paper (or a scale) on which the span and centre line have been marked off, keeping the centre line in the neighbourhood of the centre of gravity of the more heavily loaded wheels of an engine. In this way the load system required for any given span can generally be seen at a glance; occasionally, however, there may be some doubt as to whether the load system selected in this way is the correct one or not; as, for instance, when by moving the paper strip a little to the right or left, so as to take in an extra load, the new load system seems almost as promising as the one first selected. In

such a case it may be necessary to determine the maximum bending moments for two (or even three) load systems, in order to get the bending moment required.

Having determined the load system for any given span, the maximum bending moment and the point at which it occurs may be determined, analytically, as shown by the following example. Let the span be 50 feet, and the type-train that represented in Fig. 72; the load system giving maximum

Fig. 72.



bending includes the wheel (or axle) loads marked a to g, and, therefore, the total load on the span is

$$W = 9 + 9 + 18\tfrac{1}{4} + 18\tfrac{1}{4} + 18\tfrac{1}{4} + 13\tfrac{3}{4} + 13\tfrac{3}{4} = 100 \cdot 25 \text{ tons.}$$

Taking moments about wheel d, and dividing by W, we obtain, as the distance of the centre of gravity of the load system to the left of d,

$$x = (18 \cdot 25 \times 7 \cdot 5 + 9 \times 14 \cdot 75 + 9 \times 21 \cdot 25 - 18 \cdot 25 \times 7 \cdot 5 - 13 \cdot 75 \times 19 \cdot 20 - 13 \cdot 75 \times 24 \cdot 70) \div 100 \cdot 25 = -2 \cdot 788 \text{ feet.}$$

The maximum bending moment usually occurs under that wheel which is nearest to the centre of gravity of the total load, and this is always the case if the wheel be d. If, however, the centre of gravity fall a little nearer c or e than d, the bending moments under both c and d, or e and d, should be

obtained, and the greater of the two values taken. In the present example, the maximum bending moment will occur under wheel d, when that wheel is at a distance of $\frac{2 \cdot 788}{2} = 1 \cdot 394$ foot to the left of the centre of the span.

The distance of the centre of gravity of the total load W_1 from the right terminal of the span = the distance of wheel d from the left terminal = 25 − 1·394 foot = 23·606 feet. Therefore, reaction at left terminal

$$= \frac{100 \cdot 25 \times 23 \cdot 606}{50} = 47 \cdot 33 \text{ tons,}$$

the algebraical sum of the moments of forces acting to the left of d, about that point—that is, the maximum bending moment required,

$$= 47 \cdot 33 \times 23 \cdot 606 - 18 \cdot 25 \times 7 \cdot 5 - 9 \times 14 \cdot 75 - 9 \times 21 \cdot 25$$
$$= 656 \cdot 4 \text{ foot-tons.}$$

and the equivalent uniform load is

$$w' = 656 \cdot 4 \times \frac{8}{50} = 105 \cdot 02 \text{ tons.}$$

In preparing tables of equivalent live loads derived from the maximum bending moments, the work may advantageously be systematically arranged in a tabular form, as shown in Tables I, II, and III. In Table I the first column

TABLE I.

Wheel Load.	Distance of Load from Wheel *d*.	+ve Moments about Wheel *d*.	Total + Moments about *d*.	Wheel Load.	Distance of Load from Wheel *d*.	−ve Moments about Wheel *d*.	Total −ve Moments about *d*.
Tons.	Feet.	Tons × Feet.	Tons × Feet.	Tons.	Feet.	Tons × Feet.	Tons × Feet.
c = 18·25	7·50	136·87	136·87	c = 18·25	7·50	136·88	136·88
b = 9·00	14·75	132·75	269·62	f = 13·75	19·21	264·00	400·88
a = 9·00	21·25	191·25	460·87	g = 13·75	24·71	339·63	740·51
i'' = 13·75	29·79	409·61	870·48	h = 13·75	30·21	415·25	1155·76
h'' = 13·75	35·29	485·37	1355·85	i = 13·75	35·71	491·88	1647·64
g'' = 13·75	40·79	561·00	1916·85	a' = 9·00	44·25	398·25	2045·89
f'' = 13·75	46·29	636·63	2553·48	b' = 9·00	50·75	456·75	2502·64
e'' = 18·25	58·00	1058·50	3611·98	c' = 18·25	58·00	1058·50	3561·14
d'' = 18·25	65·50	1195·38	4807·36	d' = 18·25	65·50	1195·38	4756·52
c'' = 18·25	73·00	1332·25	6139·61	e' = 18·25	73·00	1332·25	6088·77

gives the loads acting to the left of wheel *d* (the wheel under which the greater number of maximum bending moments occur), the second column their distances from that load, the third column the products of load × distance, and the fourth columns the sums of these products. The fifth, sixth, seventh, and eighth columns respectively give similar values for the loads acting to the right of *d*.

In the first column of Table II the spans are set down, and in the second

TABLE II.

Span.	Load System giving Maximum Bending.	Moment (M) of Load System about Wheel *d*.	Total Load (W).	Distance $\left(\frac{M}{W}\right)$ of centre of Gravity of Load System to Left of		Distance of Centre of Span, for Maximum Bending, to Left of		
				Wheel *d*.	Wheel *e*.	Wheel *d*.	Wheel *e*.	Centre of Gravity of Load System.
Feet.		Tons × Feet.	Tons.	Feet.	Feet.	Feet.	Feet.	Feet.
20	*c* to *e*	0·00	54·75	0·00	..	0·00	..	0·00
30	*b* ,, *e*	+ 132·75	63·75	+ 2·08	..	1·04	..	− 1·04
40	*b* ,, *f*	− 131·25	77·50	− 1·69	..	− 0·85	..	+ 0·85
50	*a* ,, *g*	− 279·64	100·25	− 2·79	..	− 1·39	..	+ 1·39
60	*a* ,, *i*	− 1186·75	127·75	− 9·29	− 1·79	..	− 0·90	+ 0·90
70	*a* ,, *i*	− 1186·75	127·75	− 9·29	− 1·79	..	− 0·90	+ 0·90
80	*h″* ,, *i*	− 291·76	155·25	− 1·88	..	− 0·94	..	+ 0·94
90	*g″* ,, *a′*	− 129·01	178·00	− 0·73	..	− 0·37	..	+ 0·37
100	*f″* ,, *a′*	+ 507·61	191·75	+ 2·65	..	+ 1·33	.	− 1·33
110	*f″* ,, *b′*	+ 50·86	200·75	+ 0·25	..	+ 0·12	..	− 0·12
120	*e″* ,, *c′*	+ 50·86	237·25	+ 0·21	..	+ 0·10	..	− 0·10
130	*d″* ,, *c′*	+ 1246·24	255·50	+ 4·88	..	+ 2·44	..	− 2·44
140	*d″* ,, *d′*	+ 50·86	273·75	+ 0·19	..	+ 0·09	..	− 0 09
150	*c″* ,, *c′*	+ 50·86	310·25	+ 0·16	..	+ 0·08	..	− 0·08

column the corresponding load systems giving maximum bending moment. In the third column the algebraic sum of the moments with respect to wheel *d* are given; these being the differences of the values given in columns 4 and 8 of Table I. Thus, for a span of 50 feet the load system includes the loads marked *a* to *g*; from Table I the sum of the moments about *d* of the loads *a* to *d*

= 460·87 foot-tons, and the sum of the moments of loads d to g = − 740·51. Therefore, algebraic sum of moments for the load system a to g = − 279·64 tons × feet.

Dividing the moments given in column 3, Table II, by the total loads given in column 4, we obtain the distances given in column 5, and are thus enabled to calculate, as shown above, the abutment reactions, the maximum bending moments, and the equivalent uniform loads given respectively in columns 3, 9, and 10 of Table III.

TABLE III.

Span (S).	Distance from Left Terminal to Point (d or e). Maximum Bending.	Reaction (R) at Left Terminal.	Moment (M_1) of reaction about Point (d or e) of Maximum Bending. d.	Moment (M_1) of reaction about Point (d or e) of Maximum Bending. e.	Load System (K) to Left of Point of Maximum Bending.	Moment (M_2) of Load System (K) about Wheel d.	Moment (M_2) of Load System (K) about Wheel e.	Maximum Bending-Moment ($M_1 - M_2$).	Equivalent Uniform Live Load $W = (M_1 - M_2)\frac{8}{S}$.
Feet.	Feet.	Tons.	Tons × Feet.	Tons × Feet.		Tons × Feet.	Tons×Feet.	Tons × Feet.	Tons.
0 to 12·8	..	..	..	..	..	..	..	..	36·50
16·69	..	..	..	..	..	..	..	92·64	44·40
20	10·00	27·375	273·75	..	c	136·87	..	136·88	54·75
30	16·04	34·087	546·79	..	b to c	269·62	..	277·17	73·91
40	19·15	37·109	710·77	..	b „ c	269·62	..	441·15	88·23
50	23·61	47·330	1117·27	..	a „ c	460·87	..	656·40	105·02
60	29·10	61·969	..	1803·64	a „ d	..	869·63	934·01	124·53
70	34·10	62·242	..	2122·76	a „ d	..	869·63	1253·14	143·22
80	39·06	75·801	2960·79	..	h'' „ c	1355·86	..	1604·92	160·49
90	44·64	88·283	3940·69	..	g'' „ c	1916·86	..	2023·83	179·90
100	51·33	98·432	5052·83	..	f'' „ c	2553·49	..	2499·34	199·95
110	55·12	100·605	5545·95	..	f'' „ c	2553·49	..	2996·46	217·92
120	60·10	118·833	7142·46	..	e'' „ c	3612·00	..	3530·46	235·36
130	67·44	132·546	8938·90	..	d'' „ c	4807·37	..	4131·53	254·25
140	70·10	137·057	9606·74	..	d'' „ c	4807·37	..	4799·37	274·25
150	75·08	155·290	11,659·17	..	c'' „ c	6139·62	..	5519·55	294·38

In the diagram, Fig. 72, the spans given in column 1, Table III, are set off as abscissæ, and the corresponding equivalent uniform loads given in column 10 as ordinates. Joining the lower extremities of consecutive ordinates by straight

lines (or drawing a fair curve through these extremities), a diagram of "equivalent uniform live loads derived from maximum bending moments" is obtained, the ordinate of which, for any span, will give the equivalent uniform load for that span.

Maximum Shears, and Equivalent Uniform Live Loads derived therefrom.—The maximum shear due to any train-load will always occur at an abutment, and when the span is fully loaded. For the type-train represented by the wheel diagram (Fig. 72) maximum shear will occur either at the left abutment when the first heavily-loaded wheel c'' is just moving off the span towards the left, or at the right abutment when wheel e' is just moving off the span towards the right. For spans up to 37·56 feet the maximum shear will occur under wheel e' when the train is moving backwards (towards the right), and for spans greater than 37·56 feet it will occur under wheel c'' when the train is moving forwards. To determine the maximum shear and the corresponding equivalent uniform load, we may proceed as follows:—First, assume the train moving forwards and the wheel c'' just leaving the span, of length L, say; then, if W_1 be the sum of the wheel (or axle) loads on the span, and M_1 the sum of their moments about c'', the reaction at left abutment will be

$$W_1 - \frac{M_1}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Now assume the train moving backwards with wheel e' at right abutment, then, if W_2 be the sum of the wheel-loads on the span, and M_2 the sum of their moments about e', the reaction at the right abutment will be

$$W_2 - \frac{M_2}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The greater of these values will be the maximum shear required, and will be equal to half the "equivalent uniform load." For example, let the span be 40 feet; then, for the first case, the load system on the span will be c'' to h'', and the total load will be $W_1 = 18\frac{1}{4} + 18\frac{1}{4} + 18\frac{1}{4} + 13\frac{3}{4} + 13\frac{3}{4} + 13\frac{3}{4} = 96$ tons, the sum of the moments will be

$M_1 = 18\frac{1}{4}\,(7{\cdot}5 + 15) + 13\frac{3}{4}\,(26{\cdot}71 + 32{\cdot}21 + 37{\cdot}71) = 1739{\cdot}29$ foot-tons,

and the reaction at the left abutment will be $96 - \dfrac{1739{\cdot}29}{40} = 52{\cdot}52$ tons.

For the second case the load system will be i to e'; the total load $W_2 = 86 \cdot 5$ tons, the sum of the moments about e', $M_2 = 1382 \cdot 4$ foot-tons, and the reaction at the right abutment $86 \cdot 5 - \frac{1382 \cdot 4}{40} = 51 \cdot 94$ tons.

Hence, $52 \cdot 52$ tons is the maximum shear required, and $52 \cdot 52 \times 2 = 105 \cdot 04$ tons is the corresponding equivalent uniform load. In preparing tables of equivalent uniform live loads derived from maximum shears, the work may be recorded as shown in Table IV, and a diagram constructed therefrom as shown in Fig. 72.

In plotting the diagram the ordinate to be taken for any given span will, of

TABLE IV.—Equivalent Uniform Live Loads derived from Maximum Shears.

For Train-Load moving thus ⟵					For Train-Load moving thus ⟶				
Span (L).	Load System.	Total Load (W) on Span.	Moment, about c'', of Total Load ÷ Span $\left(\frac{M}{L}\right)$	Equivalent Uniform Load $= 2\left(W - \frac{M_1}{L}\right)$	Span (L).	Load System.	Total Load (W) on Span.	Moment, about e', of Total Load ÷ Span $\frac{M}{L}$.	Equivalent Uniform Load $= 2\left(W - \frac{M}{L}\right)$
Feet.		Tons.	Tons.	Tons.	Feet.		Tons.	Tons.	Tons.
0 to 7·5	..	..	..	36·50	0 to 7·5	..	..	..	36·50
10	c'' & d''	36·50	13·69	45·62	10	d' and e'	36·50	13·69	45·62
15	c'' & d''	36·50	9·12	54·75	15	d' and e'	36·50	9·12	54·75
20	c''—e''	54·75	20·53	68·44	20	c'—e'	54·75	26·53	68·44
25	c''—e''	54·75	16·42	76·65	25	b'—e'	63·75	24·44	78·62
30	c''—f''	68·50	25·93	85·14	30	a'—e'	72·75	28·99	87·52
35	c''—g''	82·25	34·88	94·74	35	a'—e'	72·75	24·85	95·80
40	c''—h''	96·00	43·48	105·04	40	i—e'	86·50	34·56	103·88
50	c''—i''	109·75	46·67	126·16	50	g—e'	114·00	52·69	122·61
60	c''—b	127·75	55·39	144·72	60	f—e'	127·75	56·24	143·02
70	c''—c	146·00	64·61	162·78					
80	c''—d	164·25	73·12	182·26					
90	c''—e	182·50	81·38	202·24					
100	c''—g	210·00	99·32	221·37	100	a—e'	200·05	90·09	220·82
110	c''—i	237·50	116·78	241·44					
120	c''—a'	246·50	115·83	261·34					
130	c''—b'	255·50	115·49	280·02					
140	c''—d'	292·00	142·52	298·96					
150	c''—e'	310·25	150·79	318·92	150	c''—e'	310·25	151·33	317·84

course, be the greater of the two values given in columns 5 and 10, Table IV. From a diagram thus obtained the equivalent uniform load for any span can be scaled off at once.

Graphical Methods.—The graphical methods devised by the author for determining the maximum bending moments and maximum shears required in the preparation of equivalent uniform loads were fully explained in Chapter IV.

In order to ascertain the degree of accuracy which could be obtained in practice by the use of these methods, and by the exercise of ordinary care and skill, a student of the author's, Mr. Robert Boyle, was set to work out, independently, the example illustrated in Fig. 68, Chapter IV. Mr. Boyle's results are given in Table V, as well as the actual values calculated by the author. The

TABLE V.

Span.	Equivalent Uniform Live Loads, in Tons, derived from						Distance of Centre of Span, for Maximum Bending, to Left of Wheel *d* or *e*.	
	Maximum Bending Moments.			Maximum Shears.				
	Mr. Boyle's Results.	Calculated Values.	Error Per Cent.	Mr. Boyle's Results.	Calculated Values.	Error Per Cent.	Mr. Boyle's Results.	Calculated Values.
Feet.							Feet.	Feet.
20	55·00	54·75	0·5	68·00	68·44	0·66	0·00	0·00
30	74·20	73·91	0·4	87·00	87·52	0·58	+ 1·00	+ 1·04
40	88·70	88·23	0·52	106·20	105·04	1·09	− 0·90	− 0·85
50	105·00	105·02	0·02	127·00	126·16	0·67	− 1·45	− 1·39
60	124·00	124·53	0·43	145·40	144·72	0·46	− 1·00 (*e*)	− 0·90 (*e*)
70	143·00	143·22	0·15	163·00	162·78	0·14	− 1·00 (*e*)	− 0·90 (*e*)
80	160·60	160·49	0·07	182·50	182·26	0·14	− 1·08	− 0·94
90	179·40	179·90	0·28	202·50	202·24	0·13	− 0·42	− 0·36
100	199·60	199·95	0·17	221·70	221·37	0·17	+ 1·17	+ 1·32
110	216·80	217·92	0·55	241·50	241·44	0·03	+ 0·12	+ 0·12
120	234·60	235·36	0·32	262·00	261·34	0·26	+ 0·17	+ 0·10
130	253·50	254·25	0·30	280·50	280·02	0·17	+ 2·50	+ 2·44
140	272·50	274·25	0·64	299·50	298·96	0·01	+ 0·12	+ 0·09
150	293·00	294·38	0·46	319·00	318·92	0·03	+ 0·15	+ 0·08

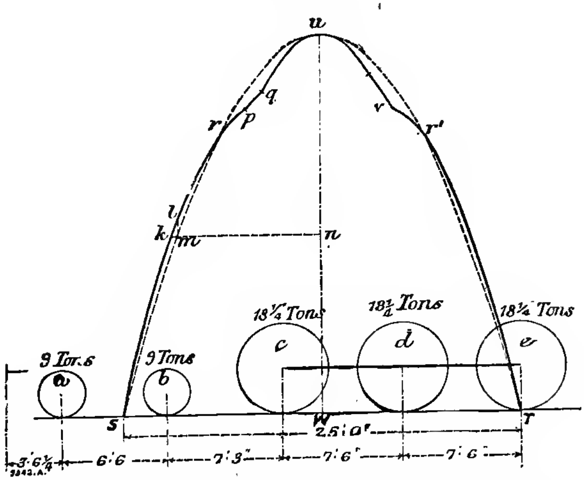
close agreement of these results with the actual values seems to show that the graphical methods are not only simple and expeditious, but fairly accurate as well.

CHAPTER VI.

Diagrams of Maximum Bending Moments.

The equivalent uniform live load derived from the greatest maximum bending moment having been determined for any given type-train and span, and the corresponding parabola or diagram of bending moments drawn, it will be found that at the haunches the maximum moments produced by the actual

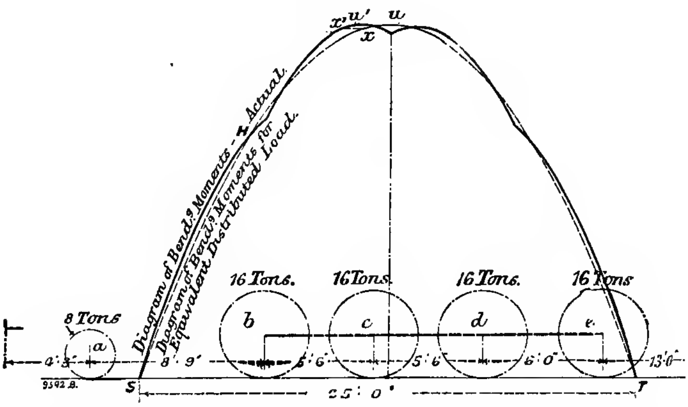
Fig. 73.



wheel-loads are somewhat greater than the moments given by the parabola. In Fig. 73, for instance, where the ordinates to the full line S k u T represent the maximum moments due to the actual wheel-loads as the six-wheel coupled engine shown moves across the span from right to left, and those to the parabola the moments due to the equivalent uniform load, it is seen that,

between r and S and r' and T, the maximum moments everywhere exceed those given by the parabola. At the distance $m\,n$ = 9 feet from the centre of the span, the excess, $m\,l$, amounts to 10 per cent. of the bending moment (98·9 tons × feet) at m, or 4·8 per cent. of that at the centre—205·3 tons × feet. Again, for the eight-wheel coupled engine, partly shown in Fig. 74, and for a like span of 25 feet, the excess is nowhere greater than about 3·5 per cent. of the maxima—225·3 tons × feet; and for the 75-foot span and train-load represented in Fig. 75 the excess at V in the bending moment diagram is only some 25 tons × feet, or less than 2 per cent. of the maximum—

Fig. 74.

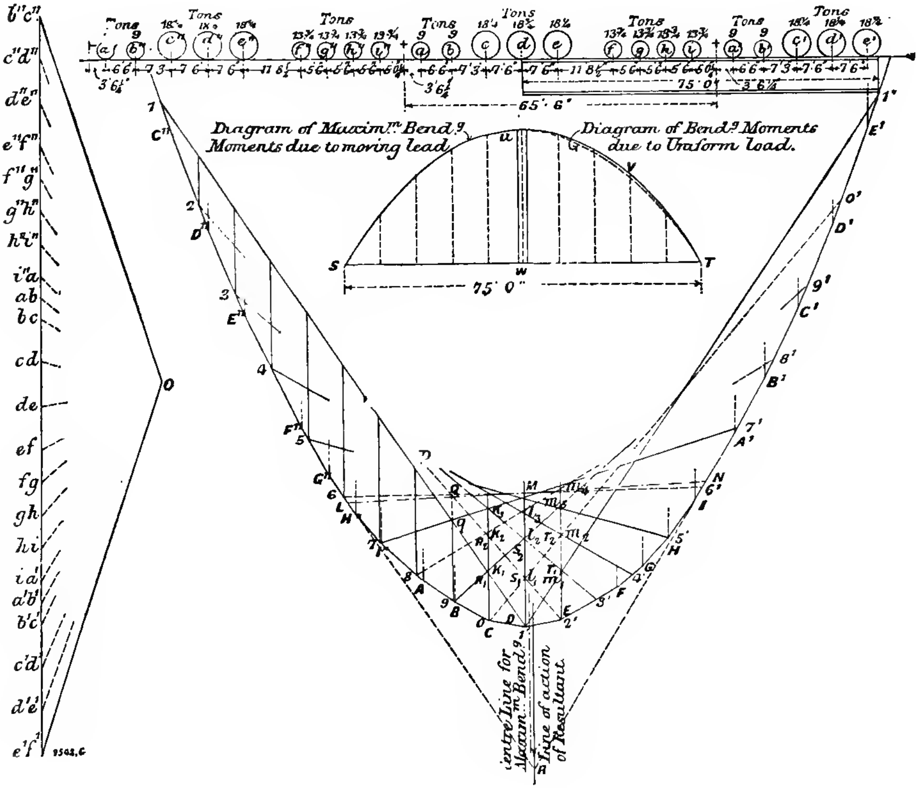


1416·5 tons × feet. This excess bending at the haunches may be provided for in practice, when the girders are designed from the equivalent uniform load, by extending the flange-plates some distance beyond the points at which, according to the parabola, they ought to stop short. For the case shown in Fig. 73 the greatest extension of flange-plates required for this purpose is that represented by Km, or about 4 inches; for the case shown in Fig. 74, between S and H, it is less than 7 inches, and for the case given in Fig. 75 it is not greater than about 10 inches.

When, however, the point at which maximum bending occurs is some distance away from the centre, the excess of the actual maximum bending

moments near that point, over and above those given by the parabola, may require a greater extension of the outer flange-plates than the 10 inches (or less) required at the haunches. For example, in Fig. 74, where the point u' of maximum bending is nearly 18 inches from the centre of the span, the extension required for the outer flange-plates may be anything up to this amount.

Fig. 75.



Again, in the case of a skew bridge the point of maximum bending may occur at a considerable distance to one side of the centre, due to the unsymmetrical manner in which it is loaded; hence, in such a case the tables of equivalent uniform loads cannot be applied, and it is therefore

necessary to draw out the actual diagram of maximum bending moments before preparing the working drawings. It may also be advisable, before preparing the working drawings, to draw out the diagram of maximum moments for bridges which are not on the skew, or where the amount of obliquity is negligibly small, and for such cases the writer has found the following methods expeditious and sufficiently accurate (when drawn out on a fairly large scale) for practical purposes.

Description of Diagram of Maximum Bending Moments (Fig. 75).—The wheel diagram and funicular polygon C″ D″ E″ . . . D′ E′ are first drawn as already explained in Chap. IV, and the diagram of maximum bending moments constructed by either of the two methods illustrated in Fig. 75.

First Method.—The girder is placed, in the first instance, with its right terminal under wheel *d*, when the corresponding diagram of bending moment will be: 1, C″, D″, E″ . . . B C 1′. Vertical lines are then drawn through *d*, and at intervals of one-tenth (or one-twentieth) of the span to the right and left of it, and some of these, cutting the diagram of bending moments, will give the intercepts 0 R_1, 9*q*, 8*p* . . ., which represent the bending moments at the respective distances from the right terminal of one-tenth, two-tenths, three-tenths the span, and so on. The girder is now advanced towards the right through a distance of one-tenth the span, when the new diagram of bending-moments will be 2 D″ E″ . . . 0 1′ 2′, and the new intercepts 1′ S_1, 0 R_2, 9 Q. . . .

Advancing the girder in this way through successive intervals of one-tenth span until the left terminal comes under *d* (or further, if the right terminal has not come under or passed the corresponding wheel *d*′ of the next engine), we obtain for each of the points taken along the span a set of different values for the bending moment, each of which should be scaled off the diagram and tabulated. The greatest maximum bending moment, D M, should also be determined, and the distance (10 inches) from the centre at which it occurs, before proceeding to plot the diagram of maximum bending moments S U T. For points at equal distances from the terminals, of course, only the greatest of the values in the two sets tabulated would be required in plotting the diagram S U T.

In the great majority of cases the bending moment at any point along the span will be a maximum for that point when the point is under one or other of the heavy wheel-loads, and when, at the same time, the train-load extends over the whole span. In order to determine the maximum moment for any given point, therefore, it will only be necessary to determine the values given as the heavy wheel-loads come successively over the point; hence the following mode of procedure.

Second Method.—Place the girder with its left terminal under wheel d (Fig. 75), obtaining, as the diagram of bending moments, the figure $1'$ E F G . . . E$'$ $1''$, and the intercept, $2' m_1$. Now advance the girder to the left until the left terminal comes under the wheel c, and obtain the new diagram of moments 0 D E F . . . D$'$ $0'$, the intercept $1' l_1$ at a distance of 7 feet 6 inches (the distance between wheels c and d) from the left terminal and the intercept $2' m_2$, at a distance of 15 feet. Advancing the girder in this way towards the left through successive intervals of 7 feet 6 inches (or half this distance if the span be a very short one, or if greater accuracy be required) until the centre of the span is reached, or passed, we obtain the moments under wheels e, d and c, represented respectively by the intercepts $2' m_1$, $1' l_1$, and $0\,K_1$ for a point 7 feet 6 inches from the left terminal, by $2' m_2$, $1' l_2$, and $0\,K_2$ for a point 15 feet from the terminal, by $2' m_3$, $1' l_3$, and $0\,K_3$ for a point 22 feet 6 inches from the terminal, and so on. The girder is now placed with its right terminal under d, and then advanced towards the right through successive intervals of 7 feet 6 inches, thus obtaining the intercepts $0\,R_1$, $1'\,S_1$, and $2'\,T_1$, as representing the bending moments under wheels c, d and e respectively for a point 7 feet 6 inches from the right terminal, the intercepts $0\,R_2$, $1'\,S_2$, and $2'\,T_2$ for a point 15 feet from the terminal, and so on. The values of these moments in tons-feet, together with the greatest maxima, D M, may then be tabulated as on page 70.

In plotting the diagram of maximum bending moments, of course, only the greatest of the six values obtained for each point—those underlined—are to be taken.

To show how the method is to be applied to the case of a train-load in which the heavy wheels of the engines are not equally spaced, as in the present

example, we will take the case of a train of the eight-wheel coupled engines partly shown in Fig. 74. In this case the moments under wheels *b*, *c* and *d* would be first determined, as above, for a series of points 5 feet 6 inches apart, and then the operation repeated for points under wheels *d* and *e* at intervals of 6 feet apart. We should thus obtain two sets of ordinates, which, being plotted, together with the ordinate representing the greatest maxima, and a fair curve

Distance from a Terminal.			Bending Moments in Ton-Feet under Load.		
Ft.	Ins.		*c.*	*d.*	*e.*
7	6	left	*537*	516	495
		right	483	508	530
15	0	left	930	*937*	895
		right	883	920	891
22	6	left	1,187	*1,225*	1,205
		right	1,183	1,180	1,166
30	0	left	1,337	1,375	*1,380*
		right	1,337	1,341	1,350
37	6	left	1,387	1,415	*1,416*
		right	1,387	1,415	1,415
Chief maxima (10 inches from centre) .				*1416·5*	

drawn through the extremities of some and slightly above those of the others, will give the maxima at every point.

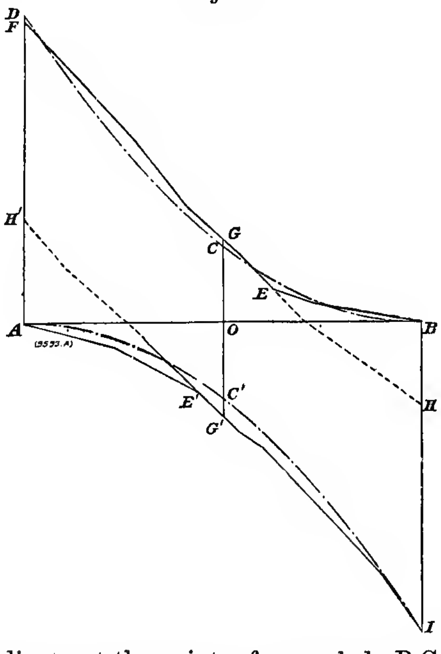
As the funicular polygon and wheel diagram are applicable to a great variety of spans, the constructions above indicated should be made on tracing-paper, or, better still, "butter" paper, as this paper is sufficiently transparent for the purpose, and presents a better surface for drawing upon than ordinary tracing paper.

CHAPTER VII.

Diagrams of Maximum Shear.

The shear at any point of a beam acted upon by a uniform moving load is a maximum when the load extends over the segment of the beam between the point and the farther abutment; and further, this maximum shear is

Fig. 76.



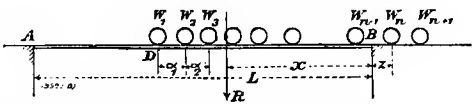
proportional to the ordinate, at the point, of a parabola B C D, or A C′ I, Fig. 76, If, for any given type-train and span, the parabola corresponding to the "equivalent uniform live load derived from the maximum shear" be drawn, the shear as given by the ordinate to this parabola for any point along the span will be found

to be less, and sometimes very considerably less, than the maximum shear produced at the point by the actual wheel-loads.

For example, in Fig. 76, the maximum positive shears produced by an engine of the type shown in Fig. 78 as it moves across the span from right to left are given by the ordinates to the full line B E G F, the maximum negative shears by the ordinates to the line A E′ G′ I, and the shears due to the "equivalent uniform load" by the ordinates to the parabolas B C D and A C′ I. In this case, therefore, we see that everywhere, except at the abutments, the actual maximum shears are greater than the shears as given by the parabola; the excess, C′ G′, at the centre amounting to about 21 per cent.

Similarly, for the type-train shown in Figs. 78 and 79, and a span of 75 feet, the diagram of maximum shears, Fig. 80, shows that the maximum shears due to the actual wheel-loads are again everywhere greater than the shears corresponding to the equivalent uniform load. In this case, at the centre of the

Fig. 77.



span, the actual maximum, when the axle-loads are taken, is equal to 24·2 tons, while the shear due to the equivalent uniform load is only 21·7 tons, or 10·3 per cent. less. It is obvious, therefore, that while tables of "equivalent uniform live loads" derived from the maximum shears may be of great value in making the preliminary calculations, it will often be necessary to draw out the actual diagrams of maximum shears, or at least determine the maximum shear at the centre and at one or two other points along the span, before preparing the working drawings.

The work of constructing diagrams of maximum shear may be considerably facilitated by noting the following observations:—In Fig. 76 the maximum positive shear for any point under B E occurs when the first wheel a is over the point, and for the remainder of the span it occurs when the first heavy wheel c is over the point. Again, assuming the engine as running backwards from left to right, the maximum negative shear for the part above A E′ occurs at any

point when the first wheel, i, is over the point, and for the remainder of the span it occurs when the wheel e is over the point. This would seem to indicate that the maximum shear at any point will occur when either a or c is over the point if the engine be running forwards, or under i or e if running backwards. This point may be investigated analytically as follows:—

To determine which of the wheel-loads at a given point D will give the maximum shear at the point, assume the train-load, in the first instance, to have moved so far along the span from the farther abutment B (Fig. 77) as to bring the front wheel W_1 over the point; and let R be the total load on the beam, and x the distance of its centre of gravity from B. The supporting force at A and shear just in front of W_1 will then be $= R.\frac{x}{L}$ (1); and so long, therefore, as no other load comes on or goes off the beam, the shear will vary as x, or according to the ordinates of a straight line.

Further, with wheel W_1 over the point D, let the loads (of which R is the resultant) on the beam be $W_1, W_2, W_3 \ldots W_{n-1}$, and let the first load off the beam be W_n and z its distance to the right of B; then, if z be less than a_1, the shear at D, as the train moves towards the left until W_2 comes over the point, will increase to

$$R\frac{x+a_1}{L} + W_n\frac{a_1 - z}{L} - W_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Of the two wheels W_1 and W_2 therefore, W_2 at D will give the greater or less shear at the point according as the expression

$$\frac{R}{L} + W_n\frac{a_1 - z}{a_1 . L} - \frac{W_1}{a_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is positive or negative.

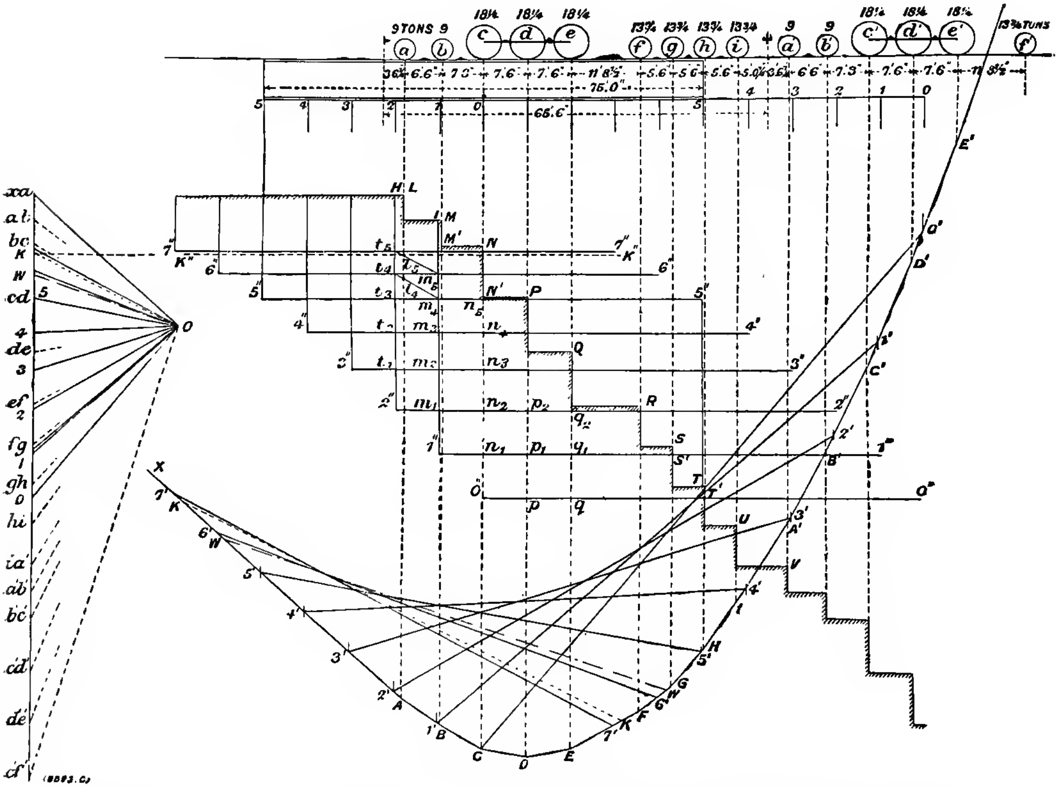
Again, with wheel W_2 over D, let R′ be the total load on the beam, and x' the distance of its centre of gravity from B; and let W_{n+1} be the first load off the beam, and z' its distance to the right of B. Then, if z' be less than (a_2), the distance between the wheels W_2 and W_3, of these two wheels at D, W_3 will give the greater or less shear according as

$$\frac{R'}{L} + W_{n+1}\frac{a_2 - z'}{a_2 . L} - \frac{W_2}{a_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is positive or negative.

Now for the train of six-wheels coupled engines shown in Figs. 78 and 79, moving from right to left, $W_1 = 9$ tons $= W_2$, $W_3 = 18\frac{1}{4}$ tons, $a_1 = 6 \cdot 5$ feet, and $a_2 = 7 \cdot 25$ feet. Hence, for this case, expressions (3) and (4) cannot be respectively positive and negative, or, in other words, W_2 can never give a

Fig. 78.



maximum shear at any point. In the same way it can easily be shown (see page 49) that W_3 (or wheel c) at D will give a greater shear at that point than will W_4, and W_4 a greater shear than W_5, and so on.

When the train shown in Fig. 78, therefore, is moving from right to left, the maximum shear at any point along the span will occur when either wheel (a

or wheel *c* is over the point; and in the same way, when the train is moving from left to right, Fig. 79, it can easily be shown that maximum shear will always occur under either wheel *i* or wheel *e*.

Diagrams of Maximum Shear Construction.

On the diagram already drawn and used in determining the greatest maximum bending moment, etc., add the load steps L M M′ N P Q . . ., Fig. 78, and then, on a sheet of tracing paper placed over the diagram, draw, at a distance apart equal to one-tenth of the span, a series of vertical lines, one of which is made to pass through the centre of wheel *c*. These lines will cut the funicular polygon in points 7′, 6′, 5′ . . . Now assume the girder placed with its left terminal below wheel *c*; the loads on the girder will then be *c*, *d*, *e* . . . *a*′, *b*′, *c*′, and *d*′, and the corresponding diagram of bending moments will be C D E . . . C′ D′ O′. Through point O in the force polygon draw O *o* parallel to the closing line C *o*′ to cut the line of loads in *o*, and through *o* draw the horizontal line *o*″ *o*″ under the girder.

The diagram of positive shears will then be *o*″ N N′ P Q . . . T T′; *o*″ N representing the shears at the left terminal, and *p* P, *q* Q . . . the shears at the distances of 0·1, 0·2, . . . of the span from that terminal. If the girder be now advanced to the left, under the loads, through a distance of one-tenth span the new diagram of bending moments will be 1′ C D E . . . B′ C′ 1′; and if a line O 1 be drawn parallel to the closing line 1′ 1′ to cut the line of loads in 1, and through this latter point a horizontal line 1″ 1″ be drawn under the girder, we shall obtain the new diagram of positive shears 1″ M M′ N . . . S S′, in which the shears at the left terminal and at the distances from it of 0·1, 0·2, 0·3 span . . . are now represented by 1″ I, n_1 N, p_1 P. . . . Advancing the girder in this way through successive intervals of one-tenth span we obtain a series of diagrams of positive shear corresponding to the base lines 0″ 0″, 1″ 1″, 2″ 2″ . . . each of which gives a value for the shear at the left terminal and at the distances of 0·1, 0·2, 0·3, 0·4, and 0·5 span from it. These values can now be scaled from the diagrams and tabulated, and the greatest of the values given for each point along the span taken and used in plotting

the diagram of maximum positive shears. A similar method may then be adopted as shown in Fig. 79, for constructing the diagram of maximum negative shears. As the maximum positive shear at any point, however, occurs when either wheel *c* or wheel *a* is over the point, and the maximum negative shear when either wheel *e* or wheel *i* is over the point, the construction of the diagram is facilitated considerably by adopting the following procedure. Under wheel *c* the shears at the left terminal, and at the distances from it of 0·1, 0·2, 0·3, 0·4, and 0·5 span, are given by $0''$ N, n_1 N, n_2 N, n_3 N, n_4 N and n_5 N respectively; or, in the force diagram, by 0–*b c*, 1–*b c*, 2–*b c*, 3–*b c*, 4–*b c*, and 5–*b c*. To determine the shears under wheel *a* we may proceed as follows:—For the position of the girder represented by the base line $6''6''$ the shear at the centre of the span is m_5 I, and for the position $7''7''$ it is t_5 H. Now during this change in the girder's position, if the total load upon the girder remained the same the shear at the centre would, according to equation (1), page 73, diminish at a uniform rate, as represented by the line m_5t_5; and, therefore, when wheel *a* was over the centre the shear at that point would be l_5 L. Similarly, at a distance of one-tenth span from the centre, when wheel *a* was over the point the shear would be l_4 L, and so on. If, however, during the change of position of the girder a new load came on, or a load went off the girder, the shears under load *a* would not be given exactly by l_5 L, l_4 L, etc.; but if the steps through which the girder is advanced were small—say, one-twentieth of the span—the errors thus introduced would be negligible.

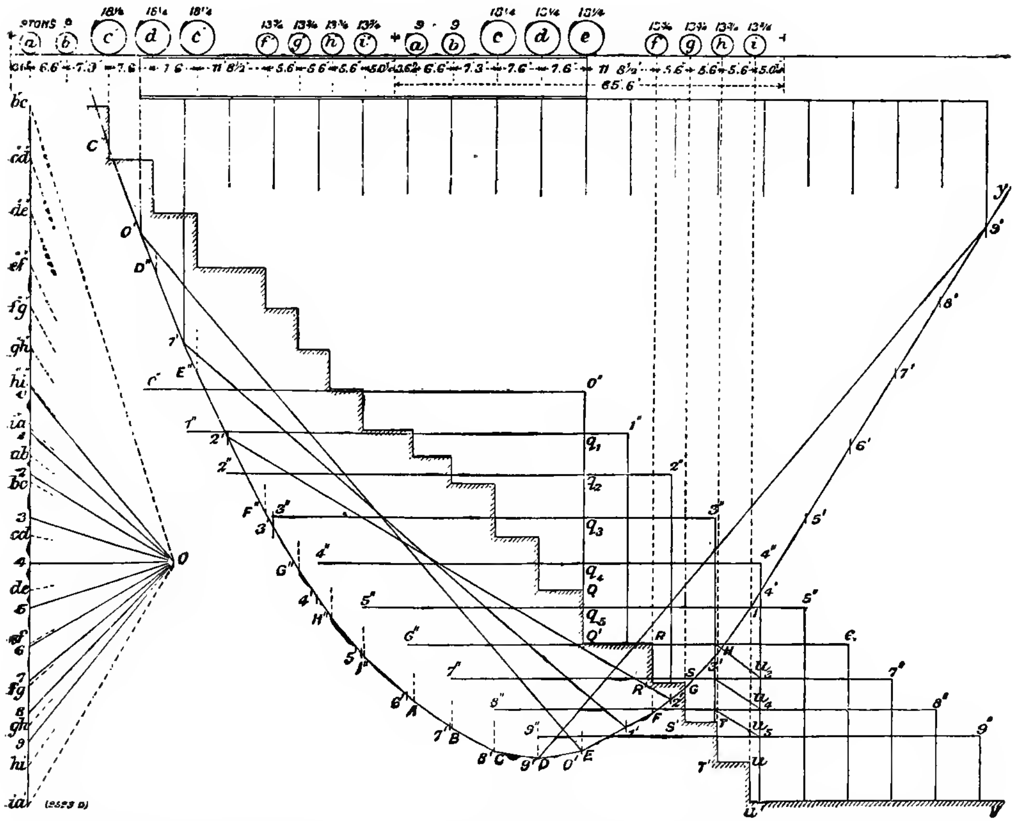
Proceeding in the same way to determine the maximum (negative) shears for the half girder on the right, the girder is first placed with its right terminal under wheel *e*, Fig. 79, and then advanced towards the right through successive intervals of one-tenth span. In this way the shears under wheel *e* are first obtained, and then those under wheel *i*, the shears under wheel *e* at the right terminal and at intervals of one-tenth span from it being given by o'' Q′, q_1 Q′, q_2 Q′ . . . q_5 Q′, and those under wheel *i* by u_5 U′ at the centre, and by u_4 U′, u_3 U′ . . . at intervals of one-tenth span from it.

For the example worked out and illustrated in Figs. 78, 79 and 80, the maximum positive shears at the left terminal and at the distances from it of 0·1, 0·2, 0·3, and 0·4 span, occur under wheel *c*, and are represented by

o'' N, n_1 N, n_2 N, n_3 N, and n_4 N respectively; at the centre, however, the positive shear is a maximum when under wheel a, and is represented by l_5 L, Fig. 78.

The maximum negative shears at the right terminal and at intervals of

Fig. 79.

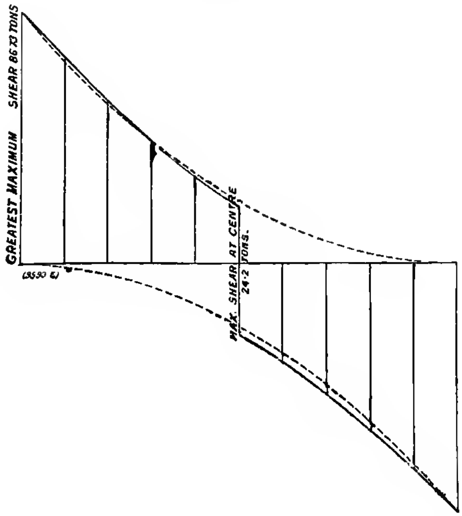


one-tenth the span from it are given by o'' Q′, q_1 Q′, q_2 Q′ (under e), u_3 U′, u_4 U′, and u_5 U′ (under i).

The maximum shears having been thus determined for a number of points along the span, the diagram can be plotted as shown in Fig. 80.

Instead of advancing the girder through successive intervals of one-tenth span, as in the example given (or of one-twentieth span, where great accuracy is required), an approximation to the true form of the diagram of maximum shears, and one which would be sufficiently accurate for the great majority of

Fig. 80.



cases met with in practice, could be obtained by determining the maximum shears at the abutments, at a quarter span, and at the centre, and then erecting ordinates representing the maximum shears at these points and joining their extremities by straight lines.